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Subharmonic Solutions of Order $\frac{1}{n}$ $(n = 2, 3)$ to a Weakly Nonlinear Second Order ODE Governed the Motion of a TM-AFM Cantilever

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ABSTRACT

In this paper, we use the method of multiple scales to investigate the perturbation analysis for a weakly nonlinear second order differential equation that governs the dynamic behavior of a micro-cantilever based on tapping mode atomic force microscopy. Furthermore, we focus on examining two distinct categories of periodic solutions, specifically subharmonic solutions with an order of $\frac{1}{n}$ $(n = 2, 3)$). For each type of solution, the frequency response equations, peak amplitudes and their positions, the steady state solutions, and the approximate analytical formulas are given together with the modulation equations of the amplitude and phase. Additionally, in order to demonstrate how the parameters affect the solution, numerical solutions of the frequency response equations and the stability conditions are performed. The results are shown in a variety of figures. Finally, there is a discussion and conclusion.

KEYWORDS

Micro-electro-mechanical system, atomic force microscopy, differential equations, subharmonic solutions, multiple scales method

1. Introduction

Atomic force microscopy (AFM), is widely regarded as a vital tool for studying material surfaces. Because AFM can assess forces in the nanoNewton range, it has numerous applications in the manipulation of carbon nanotubes, nanolithography, data-storage technologies, and semiconductor devices [1, 12]. Most classical dynamical systems as well as nonclassical dynamical systems (such as Micro and Nano-electro-mechanical systems, or MEMS/NENS) can be studied theoretically to produce nonlinear second order ordinary differential equations(ODEs) or a set of nonlinear coupled second order ODEs [8, 9, 14–16]. Thus, significant research has been done to investigate the several periodic solutions (harmonic, sub, super, sub-super, super-sub, and combinations of harmonic solutions) of these ordinary differential equations using perturbation analysis. For instance, Elnaggar and Alhanadwah [3] presented the subharmonic solution of order one-half for a single degree of freedom (SDOF) system with quadratic, cubic, and quartic nonlinearities under parametric excitations. In addition, Yu-Xiu and Wen-bo [18] studied analytically a 1/3 subharmonic solution for the Duffing equation. Moreover, the subharnonic solutions and the stability for a weakly damped nonlin-

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ear quasi-periodic Mathieu equation were investigated by Guennoun and *et al.* [11]. Dunne [2] discussed the subharmonic solution of a nonlinear SDOF oscillator was driven by periodic excitation. Additionally, Shooshtari and Pasha Zanoosi [17] represented the subharmonic solutions of second order weakly nonlinear ODE that represents the vibration of a mass grounded system which includes two linear and nonlinear springs in series. Mahmoodi *et al.* [13] studied the subharmonic solutions of the governing equation of the nonlinear flexural vibrations of piezoelectrically actuated microcantilevers performed. In addition, Fahsi and Belhaq [10] investigated analytically the subharmonic solutions in a self-excited parametrically forced oscillator with quadratic nonlinearity. Elnaggar *et al.* studied the subharmonic solutions of a Van der Pol equation subjected to weakly nonlinear parametric and forcing excitations [4]. Moreover, the presented the subharmonic solution of a MEMS subjected to external and parametric excitations. Additionally, Elnaggar and Khalil [7] presented the subharmonic solution for nonlinear SDOF system with two distinct time-delays under an external excitation and the subharmonic solutions of even order $(\frac{1}{2}, \frac{1}{4})$ to a weakly nonlinear second order ODE governed the motion of a MEMS were investigated by Elnaggar *et al.* [6].

This article's major focus is on the subharmonic solution of orders $(\frac{1}{2}, \frac{1}{3})$. The approximate solutions are obtained through the use the multiple scales method (MMS). The stability criteria for the steady state solutions are identified for each type of periodic solution. Further, a numerical analysis is conducted to examine the frequency response equations and the influence specific system parameters on each of these solutions. Additionally, a commentary on the figures is given.

2. Perturbation Analysis

Consider the following nonlinear second order ODE

$$u''+\zeta u'+u+\beta u^3=-\frac{d}{(\alpha+u)^2}+\frac{d\Sigma^6}{30(\alpha+u)^8}+\epsilon\left(f\text{Cos}\Omega t-\frac{\eta}{(\alpha+u)^3}u'\right), \qquad (1)$$

where $\alpha,\ \beta,\ d,\ \Sigma,\ \zeta,\eta,\ \Omega$ and f are constants; $\epsilon << 1$. Eq.(1) represents the mathematical model of the dynamic behavior of a microcantilever-based TM-AFM with squeeze film damping effects [19]. Wen-Ming Zhang *et al.* [19] solved Eq.(1) numerically by using the 4^{th} order Runge-Kutta method to investigate the characteristic and nonlinear dynamics of a TM-AFM cantilever system. Utilizing Taylor expansion, retained only three terms of expansion and applying the perturbation technique while maintaining the nonlinear terms of $O(\epsilon)$ and the amplitude of the excitation force of $O(1)$, then we get the following weakly nonlinear second order ODE

$$u''+\omega_0^2u+\epsilon(2\mu u'-\alpha_3u^2+\beta u^3-\alpha_5uu'+\alpha_6u^2u')=\alpha_1+f\text{Cos}\Omega t, \qquad (2)$$

where $\omega_0^2=1-\alpha_2, 2\mu=\zeta+\alpha_4=\zeta+\frac{\eta}{\alpha^3}, \alpha_1=\frac{d\Sigma^6}{30\alpha^8}-\frac{d}{\alpha^2}, \alpha_2=\frac{2d}{\alpha^3}-\frac{4d\Sigma^6}{15\alpha^9}$,
$\alpha_3=\frac{6d\Sigma^6}{5\alpha^{10}}-\frac{3d}{\alpha^4}, \alpha_4=\frac{\eta}{\alpha^3}, \alpha_5=\frac{3\eta}{\alpha^4}$ and $\alpha_6=\frac{6\eta}{\alpha^5}$.

An approximate solution of Eq.(2) can be obtained by a number of perturbation techniques (Nayfeh [14, 14]). According to MMS, the scaled times T_n can be introduced as:

$$T_n = \epsilon^n t, \qquad n = 0, 1, 2, ... \tag{3}$$

and differentiation with respect to the dimensionless time t, we obtain

$$\frac{d}{dt} = D_0 + \epsilon D_1 + ... \qquad \& \qquad \frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + ..., \tag{4}$$

where $D_n = \frac{\partial}{\partial T_n}$. Now, we assume a two scale expansion for the solution of Eq. (2) in the form

$$u(t;\epsilon) = u_0(T_0, T_1) + \epsilon u_1(T_0, T_1) + ... \tag{5}$$

and substituting from Eqs.(4) and (5) into Eq.(2), then equating the coefficients of the same powers of ϵ to zero, we obtain a set of linear partial differential equations

$$D_0^2 u_0 + \omega_0^2 u_0 = \alpha_1 + f\text{Cos}\Omega t, \tag{6}$$

$$\begin{aligned} D_0^2 u_1 + \omega_0^2 u_1 = & -2\mu D_0 u_0 - 2D_1 D_0 u_0 - \beta u_0^3 \\ & + \alpha_3 u_0^2 + \alpha_5 u_0 D_0 u_0 - \alpha_6 u_0^2 D_0 u_0. \end{aligned} \tag{7}$$

Solving Eq.(6) for $u_0(T_0, T_1)$, we have

$$u_0(T_0, T_1) = A(T_1) e^{i\omega_0 T_0} + \bar{A}(T_1) e^{-i\omega_0 T_0} + \kappa + \Lambda \text{Cos}(\Omega T_0), \tag{8}$$

where $i^2 = -1$, $\bar{A}$ is the complex conjugate of A, $\kappa = \frac{\alpha_1}{\omega_0^2}$ and $\Lambda = \frac{f}{\omega_0^2 - \Omega^2}$. Thus, Eq.(7) becomes

$$\begin{aligned} {D_0}^2 u_1 + \omega_0^2 u_1 = & -\frac{1}{2} e^{iT_0\omega_0} (A(i\omega_0 \left(\alpha_6 \left(2A\bar{A} + 2\kappa^2 + \Lambda^2\right) - 2\alpha_5\kappa + 4\mu\right) \\ & + 6A\beta\bar{A} - 4\alpha_3\kappa + 6\beta\kappa^2 + 3\beta\Lambda^2) + 4i\omega_0 A') \\ & - \frac{1}{8}\Lambda e^{iT_0\Omega} (i\alpha_6\Omega \left(8A\bar{A} + 4\kappa^2 + \Lambda^2\right) \\ & + 24A\beta\bar{A} - 4i\alpha_5\kappa\Omega - 8\alpha_3\kappa + 12\beta\kappa^2 + 3\beta\Lambda^2 + 8i\Omega\mu) \\ & - \frac{1}{2}\Lambda\bar{A} e^{iT_0(\Omega-\omega_0)} \left(i\left(\alpha_5 - 2\alpha_6\kappa\right)(\omega_0 - \Omega) - 2\alpha_3 + 6\beta\kappa\right) \\ & - \frac{1}{4}\Lambda^2\bar{A} e^{iT_0(2\Omega-\omega_0)} \left(3\beta + i\alpha_6\left(2\Omega - \omega_0\right)\right) \\ & - \frac{1}{2}\Lambda\bar{A}^2 e^{iT_0(\Omega-2\omega_0)} \left(3\beta + i\alpha_6\left(\Omega - 2\omega_0\right)\right) \\ & - \frac{1}{4}\Lambda^2 e^{2iT_0\Omega} \left(-i\Omega\left(\alpha_5 - 2\alpha_6\kappa\right) - \alpha_3 + 3\beta\kappa\right) \\ & - \frac{1}{8}\Lambda^3 e^{3iT_0\Omega} \left(\beta + i\alpha_6\Omega\right) + NST + cc, \end{aligned} \tag{9}$$

where NST denotes the terms does not produce secular terms and cc denotes the complex conjugate.

3. Subharmonic Solution of Order 1/2

We can get the subharmonic solution of order 1/2 (the periodic solution has its least 2 of the period of the external excitation), by putting $\Omega \approx 2\omega_0$ or

$$\Omega = 2\omega_0 + \epsilon\sigma. \tag{10}$$

Eliminating the secular terms from Eq.(9) yields

$$\begin{aligned} &A\left(-3\beta\left(2\kappa^2+\Lambda^2\right)+4\kappa\alpha_3-i\left(4\mu-2\kappa\alpha_5+\left(2\kappa^2+\Lambda^2\right)\alpha_6\right)\omega_0\right)\ -4i\omega_0 A' \\ &-2\bar{A}\ A^2\left(3\beta+i\alpha_6\omega_0\right)-\bar{A}\Lambda e^{i\sigma T_1}\left(6\beta\kappa-2\alpha_3-i\left(\alpha_5-2\kappa\alpha_6\right)\left(\Omega-\omega_0\right)\right)=0 \end{aligned} \tag{11}$$

Eq.(11) is a differential equation in complex form. To solve it, $A(T_1)$ can be expressed in polar form as:

$$A = \frac{1}{2}a(T_1)e^{i\phi(T_1)}, \tag{12}$$

where a and ϕ are real functions of T_1. Using Eq.(12) into Eq.(11) and separating real and imaginary parts, we get the following modulation equations

$$\begin{aligned} a' &= -\frac{a\left(\alpha_6\left(a^2+4\kappa^2+2\Lambda^2\right)+K_1\right)}{8}+\frac{aK_3\cos(\gamma)(\Omega-\omega_0)}{4\omega_0}-\frac{aK_2\sin(\gamma)}{8\omega_0} \\ a\gamma' &= \frac{a\left(-3a^2\beta+K_4+4\sigma\omega_0\right)}{4\omega_0}-\frac{aK_3\sin(\gamma)(\Omega-\omega_0)}{4\omega_0}-\frac{aK_2\cos(\gamma)}{4\omega_0}, \end{aligned} \tag{13}$$

where $\gamma = \sigma \mathrm{T}_1 - 2\phi$, $K_1 = 8\mu - 4\alpha_5\kappa$, $K_2 = 4\Lambda(3\beta\kappa - \alpha_3)$, $K_3 = 2\Lambda(\alpha_5 - 2\alpha_6\kappa)$ and $K_4 = 8\alpha_3\kappa - 12\beta\kappa^2 - 6\beta\Lambda^2$.

Moreover, the analytical expression of the subharmonic solution of order 1/2 can be approximated as

$$u = a\cos\frac{1}{2}[\Omega t-\gamma]+\frac{f}{\omega_0^2-\Omega^2}\cos[\Omega t]+\frac{\alpha_1}{\omega_0^2}+o(\epsilon), \tag{14}$$

where a and γ are the amplitude and phase are given by the system (13).

We can get the steady state solutions by putting $a' = \gamma' = 0$ in the system (13), we obtain

$$\begin{aligned} K_3\cos(\gamma)\left(\Omega-\omega_0\right)-\omega_0\left(\alpha_6\left(a^2+4\kappa^2+2\Lambda^2\right)+K_1\right) &= K_2\sin(\gamma) \\ -3a^2\beta-K_3\sin(\gamma)\left(\Omega-\omega_0\right)+K_4+4\sigma\omega_0 &= K_2\cos(\gamma). \end{aligned} \tag{15}$$

Squaring and adding both equations in the system (15), we get the frequency response equation as follows

$$\begin{aligned} &a^4\left(\alpha_6^2\omega_0^2+9\beta^2\right)+a^2(2K_2\left(\alpha_6\omega_0\sin(\gamma)+3\beta\cos(\gamma)\right) \\ &+2\omega_0\left(\alpha_6\omega_0\left(2\alpha_6\left(2\kappa^2+\Lambda^2\right)+K_1\right)-12\beta\sigma\right)-6\beta K_4) \\ &+2K_2\omega_0\sin(\gamma)\left(2\alpha_6\left(2\kappa^2+\Lambda^2\right)+K_1\right) \\ &+\omega_0^2\left(\left(2\alpha_6\left(2\kappa^2+\Lambda^2\right)+K_1\right)^2+16\sigma^2\right) \\ &+\cos(\gamma)\left(-8K_2\sigma\omega_0-2K_2K_4\right) \\ &+8K_4\sigma\omega_0-K_3^2\left(\Omega-\omega_0\right)^2-K_2^2+K_4^2=0. \end{aligned} \tag{16}$$

Solving Eq.(16) for σ, we obtain

$$\begin{aligned} \sigma =& \frac{3a^2\beta+K_2\cos(\gamma)-K_4}{4\omega_0} \\ &\pm\frac{\sqrt{-\omega_0^2\left(K_5+K_2\sin(\gamma)+K_3\Omega\right)\left(K_5+K_2\sin(\gamma)-K_3\Omega\right)}}{4\omega_0^2}, \end{aligned} \tag{17}$$

where $K_5 = \omega_0(\alpha_6 \left(a^2 + 4\kappa^2 + 2\Lambda^2\right) + K_1 + K_3)$.
Thus, the peak amplitude a_p would be verified in the following equation

$$\begin{aligned}&\left(\omega_0 \left(\alpha_6 \left(a_p^2 + 4\kappa^2 + 2\Lambda^2\right) + K_1 - K_3\right) + K_2 \sin(\gamma) + K_3\Omega\right)\\&\left(\omega_0 \left(\alpha_6 \left(a_p^2 + 4\kappa^2 + 2\Lambda^2\right) + K_1 + K_3\right) + K_2 \sin(\gamma) - K_3\Omega\right) = 0.\end{aligned} \tag{18}$$

Moreover, the corresponding value of σ_p is given by

$$\sigma_p = \frac{3a^2\beta + K_2 \cos(\gamma) - K_4}{4\omega_0}. \tag{19}$$

The stability of subharmonic solutions of order 1/2 can be examined by introducing a small perturbation to the steady state solutions, i. e. putting

$$a = a_0 + a_1, \tag{20}$$

$$\gamma = \gamma_0 + \gamma_1, \tag{21}$$

where a_0 and γ_0 represent the steady state solutions, a_1 and γ_1 represent the perturbation. Substituting Eqs.(20) and (21) into the system (13) and using the steady state conditions while maintaining the linear terms, we obtain

$$\begin{aligned}a_1' &= \frac{a_0\gamma_1(3a_0^2\beta - 8\alpha_3\kappa + 6\beta(2\kappa^2 + \Lambda^2) - 4\sigma\omega_0)}{8\omega_0} - \frac{1}{4}a_0^2 a_1\alpha_6,\\ \gamma_1' &= -\frac{1}{4}\gamma_1 \left(\alpha_6 \left(a_0^2 + 4\kappa^2 + 2\Lambda^2\right) - 4\alpha_5\kappa + 8\mu\right) - \frac{3a_0a_1\beta}{2\omega_0}.\end{aligned} \tag{22}$$

Substituting $a_1 = \Gamma_1 e^{\theta T_1}$ and $\gamma_1 = \Gamma_2 e^{\theta T_1}$ into the system (22), we get

$$\begin{aligned}6a_0\beta\Gamma_1 + \omega_0\alpha_6 \left(a_0^2 + 4\kappa^2 + 2\Lambda^2\right) - 4\alpha_5\kappa + 4(\theta + 2\mu)\Gamma_2 &= 0\\ 2\omega_0 \left(\alpha_6 a_0^2 + 4\theta\right)\Gamma_1 + K_6\Gamma_2 &= 0,\end{aligned} \tag{23}$$

where $K_6 = a_0\sigma\omega_0 - a_0 \left(3a_0^2\beta - 8\alpha_3\kappa + 6\beta \left(2\kappa^2 + \Lambda^2\right)\right)$.
For the nontrivial solution, the determinant of the coefficient matrix for Γ_1 and Γ_2 must vanish, which leads to a quadratic equation for the eigenvalue θ

$$\begin{aligned}\theta = &-\frac{1}{4}\left(\alpha_6 \left(a_0^2 + 2\kappa^2 + \Lambda^2\right) - 2\alpha_5\kappa + 4\mu\right)\\ &\pm \frac{\sqrt{\omega_0^4 \left(\alpha_6 \left(2\kappa^2 + \Lambda^2\right) - 2\alpha_5\kappa + 4\mu\right)^2 + 3a_0\beta K_6\omega_0^2}}{4\omega_0^2}.\end{aligned} \tag{24}$$

Therefore, the stability of the subharmonic solution can be examined by evaluating the sign of the real parts of the eigenvalues. Thus, the solution is asymptotically stable if the real parts of both eigenvalues of equation (24) are less than zero.

4. Subharmonic Solution of Order 1/3

In this case $\Omega \approx 3\omega_0$, then we can write

$$\Omega = 3\omega_0 + \epsilon\sigma \tag{25}$$

and

$$\left(\Omega - 3\omega_0\right) T_0 = \omega_0 T_0 + \epsilon\sigma T_0 = \omega_0 T_0 + \sigma T_1. \tag{26}$$

By eliminating the secular terms from Eq.(9), we get

$$\begin{aligned} &A\left(-3\beta\left(2\kappa^2+\Lambda^2\right)+4\kappa\alpha_3 - i\left(4\mu - 2\kappa\alpha_5 + \left(2\kappa^2+\Lambda^2\right)\alpha_6\right)\omega_0\right) \\ &- 2A^2\left(3\beta + i\alpha_6\omega_0\right)\bar{A} - e^{iT_1\sigma}\Lambda\left(3\beta + i\alpha_6\left(\Omega - 2\omega_0\right)\right)\bar{A}^2 - 4i\omega_0 A' = 0. \end{aligned} \tag{27}$$

Using Eq.(12) into Eq.(27) and separating real and imaginary parts, we obtain the following modulation equations:

$$\begin{aligned} a' &= -\frac{3a^2\beta\Lambda\sin(\gamma)}{8\omega_0} - \frac{a(\alpha_6(a^2+4\kappa^2+2\Lambda^2)+K_7)}{8} - \frac{a^2K_8\cos(\gamma)}{8\omega_0}, \\ \gamma' &= \frac{3a^2K_8\sin(\gamma)}{8\omega_0} - \frac{a(\beta(a^2+36\kappa^2+18\Lambda^2)-24\alpha_3\kappa-8\sigma\omega_0)}{8\omega_0} - \frac{9a^2\beta\Lambda\cos(\gamma)}{8\omega_0}, \end{aligned} \tag{28}$$

where $\gamma = \sigma T_1 - 3\phi, K_7 = 8\mu - 4\alpha_5\kappa$ and $K_8 = \alpha_6\Lambda\left(\Omega - 2\omega_0\right)$.

Moreover, the approximate analytical expression of the subharmonic solution of order 1/3 is

$$u = a\cos\frac{1}{3}[\Omega t - \gamma] + \frac{f}{\omega_0^2 - \Omega^2}\cos[\Omega t] + \frac{\alpha_1}{\omega_0^2} + o(\epsilon), \tag{29}$$

where a and γ are the amplitude and phase are given by the system (28).

Additionally, we can get the steady state solution by substituting $a' = \gamma' = 0$ in the system (28), we obtain

$$\begin{aligned} -3\omega_0\left(\alpha_6\left(a^2+4\kappa^2+2\Lambda^2\right)+K_7\right) - aK_9\sin(\gamma) &= 3aK_8\cos(\gamma), \\ 8\sigma\omega_0 - \beta\left(a^2+36\kappa^2+18\Lambda^2\right) + K_9 + aK_9\cos(\gamma) &= -3aK_8\sin(\gamma), \end{aligned} \tag{30}$$

where $K_9 = 9\beta\lambda$ and $K_{10} = 24\kappa\alpha_3$. Squaring both equations in the system (30) and adding them, we get the frequency response equation

$$\begin{aligned} &a^4\left(9\alpha_6^2\omega_0^2+\beta^2\right) + 2a^3K_9\left(3\alpha_6\omega_0\sin(\gamma)+\beta\cos(\gamma)\right) \\ &+ a^2(2(-8\beta\sigma\omega_0 + 9\alpha_6\omega_0^2\left(2\alpha_6\left(2\kappa^2+\Lambda^2\right)+K_7\right) + \beta(18\beta\left(2\kappa^2+\Lambda^2\right) \\ &- K_{10})) - 9K_8^2 + K_9^2) + a(6K_9\omega_0\sin(\gamma)\left(2\alpha_6\left(2\kappa^2+\Lambda^2\right)+K_7\right) \\ &+ 2K_9\cos(\gamma)\left(18\beta\left(2\kappa^2+\Lambda^2\right) - K_{10} - 8\sigma\omega_0\right)) + 324\beta^2\left(2\kappa^2+\Lambda^2\right)^2 \\ &+ \omega_0^2\left(9\left(2\alpha_6\left(2\kappa^2+\Lambda^2\right)+K_7\right)^2 + 64\sigma^2\right) \\ &+ 16\sigma\omega_0\left(K_{10} - 18\beta\left(2\kappa^2+\Lambda^2\right)\right) + K_{10}\left(K_{10} - 36\beta\left(2\kappa^2+\Lambda^2\right)\right) = 0. \end{aligned} \tag{31}$$

Solving Eq.(31) for σ, we obtain

$$\sigma = \frac{\beta\left(a^2+36\kappa^2+18\Lambda^2\right)+aK_9\cos(\gamma)-K_{10}}{8\omega_0} \pm \frac{\sqrt{\omega_0^2\left(9a^2K_8^2-\left(3\omega_0\left(\alpha_6\left(a^2+4\kappa^2+2\Lambda^2\right)+K_7\right)+aK_9\sin(\gamma)\right)^2\right)}}{8\omega_0^2}. \tag{32}$$

So, the peak amplitude a_p could be verified by the following equation

$$9a_p^2K_8^2-\left(3\omega_0\left(\alpha_6\left(a_p^2+4\kappa^2+2\Lambda^2\right)+K_7\right)+a_pK_9\sin(\gamma)\right)^2=0. \tag{33}$$

In addition, the corresponding value of σ_p is given by

$$\sigma_p=\frac{\beta\left(a_p^2+36\kappa^2+18\Lambda^2\right)+a_pK_9\cos(\gamma)-K_{10}}{8\omega_0}. \tag{34}$$

Furthermore, the stability of the subharmonic solutions of order 1/3 can be examined by introducing a small perturbation to the steady state solutions similar to Eqs.(20) and (21). So, we get

$$\begin{aligned} a_1' &= \tfrac{1}{8}K_{12}a_1+\tfrac{a_0K_{11}}{24\omega_0}\gamma_1, \\ \gamma_1' &= \tfrac{(K_{11}-18a_0^2\beta)}{8a_0\omega_0}a_1-\tfrac{3}{8}\left(2\alpha_6a_0^2+K_{12}\right)\gamma_1, \end{aligned} \tag{35}$$

where $K_{11}=9a_0^2\beta-24\alpha_3\kappa+18\beta\left(2\kappa^2+\Lambda^2\right)-8\sigma\omega_0$ and $K_{12}=-4\alpha_5\kappa+8\mu+\alpha_6\left(-a_0^2+4\kappa^2+2\Lambda^2\right)$. Substituting by $a_1=\Gamma_1e^{\theta T_1}$ and $\gamma_1=\Gamma_2e^{\theta T_1}$ into system (35), we get

$$\begin{aligned} 3\omega_0\left(K_{12}-8\theta\right)\Gamma_1+a_0K_{11}\Gamma_2 &= 0, \\ \left(\tfrac{K_{11}}{a_0\omega_0}-\tfrac{18a_0\beta}{\omega_0}\right)\Gamma_1+\left(-6\alpha_6a_0^2-8\theta-3K_{12}\right)\Gamma_2 &= 0. \end{aligned} \tag{36}$$

For the nontrivial solution, the determinant of the coefficient matrix for Γ_1 and Γ_2 must vanish, which leads to a quadratic equation for the eigenvalue θ.

$$\theta=\frac{1}{24}\left(-9\alpha_6a_0^2-3K_{12}\right) \pm \frac{\sqrt{3}}{24\omega_0^2}\sqrt{3\omega_0^4\left(3\alpha_6a_0^2+2K_{12}\right)^2+K_{11}\omega_0^2\left(K_{11}-18a_0^2\beta\right)}. \tag{37}$$

Therefore, he stability of the subharmonic solution of order 1/3 can be examined by evaluating the sign of the real part of the eigenvalues. Consequently, the solution is stable if the real parts of both eigenvalues of equation (37) are less than zero.

5. Numerical Results and Discussion

This section investigates the numerical results in the form of the frequency response curves obtained by solving the frequency response equations (16) and (31) while maintaining the stability conditions (24) and (37). The numerical results are plotted in groups of figures (1-7) and (8-14), which explain the variation of the amplitude a with the detuning parameter σ for a given values of the other parameters where the solid lines represent stable solutions and the dashed lines represent unstable solutions.

Figures (1-7) represent the frequency response curves for the subharmonic solution of order 1/2 for certain values of the parameters $\alpha = 0.9, \beta = 0.4, \eta = 0.006, f = 0.5, \Sigma = 0.2, \zeta = 0.001, d = 4/27$ and $\gamma = 90$.

Fig.1 shows the variation of the amplitude of the steady state solutions for different values of α. It can be seen from this figure that by decreasing α, the curves bent to the right of the σ axis and there exist two solutions; one of them is stable and the other is unstable.

From Fig.2, we note that by increasing β, the curves bent to the R.H.S and we have two solutions; one of them is stable and the other is unstable.

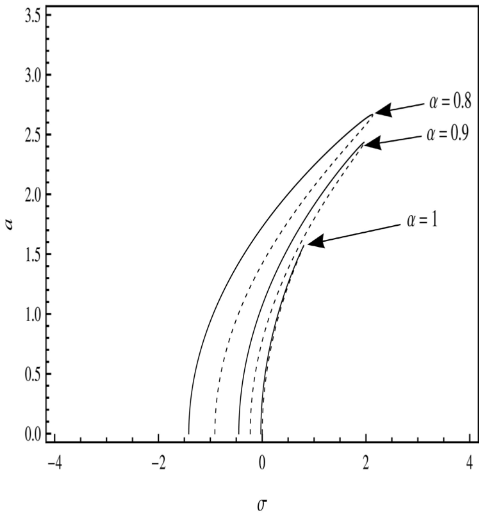

Figure 1. ***The frequency response curves for different values of*** α

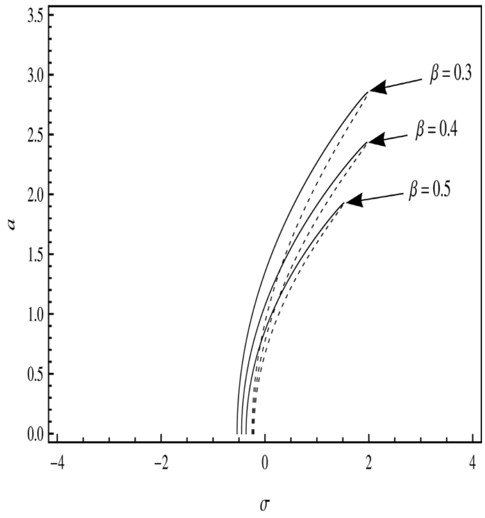

Figure 2. ***The frequency response curves for different values of*** β

From Fig.3 by decreasing η, it can be seen that the inclination in the R.H.S and there exist two solutions; one of them is stable and the other is unstable.

Fig.4 shows that by increasing f, we have two solutions; one of them is stable and the other is unstable and the bend in the R.H.S.

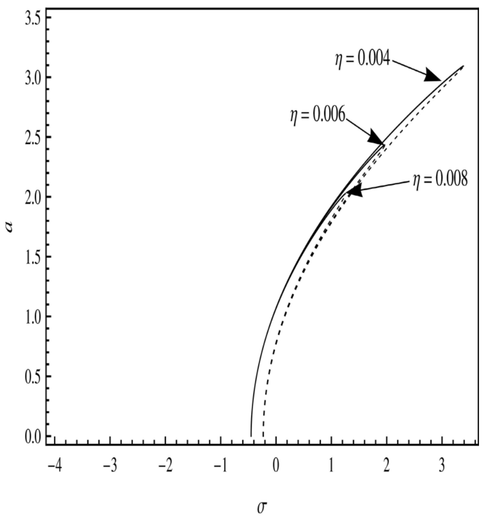

Figure 3. ***The frequency response curves for different values of*** η

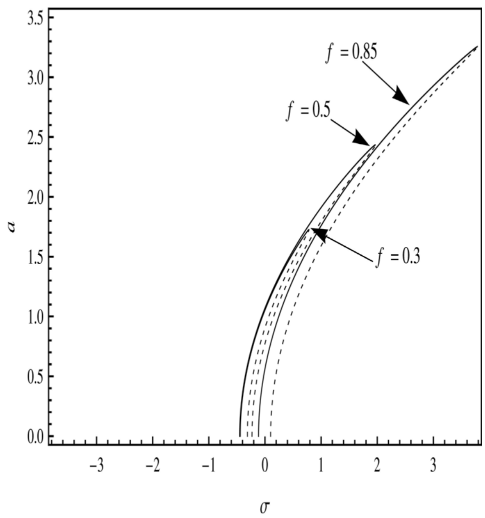

Figure 4. ***The frequency response curves for different values of*** f

From Fig.5, we observe that by decreasing Σ, the curves bent to the right of the σ axis and there exist two solutions, one of them is stable and the other is unstable.

Fig.6 shows that by decreasing ζ, the branches of the figure are unchanged. Also, we obtain two solutions; one of them is stable and the other is unstable and the bend in the R.H.S.

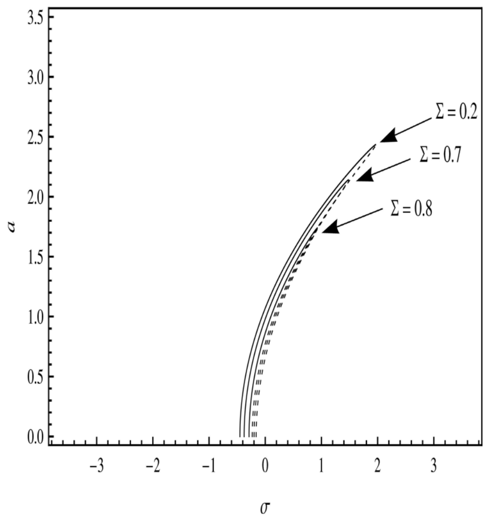

Figure 5. ***The frequency response curves for different values of*** Σ

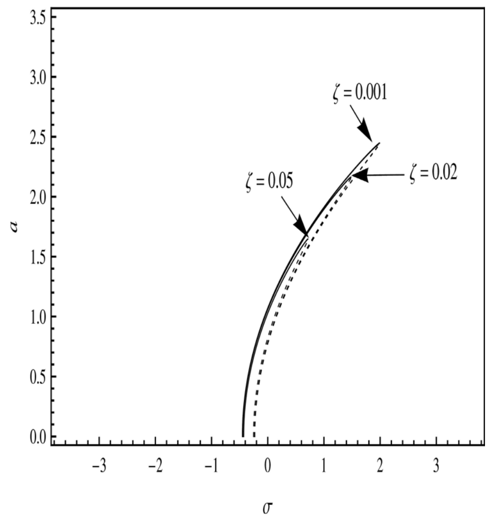

Figure 6. ***The frequency response curves for different values of*** ζ

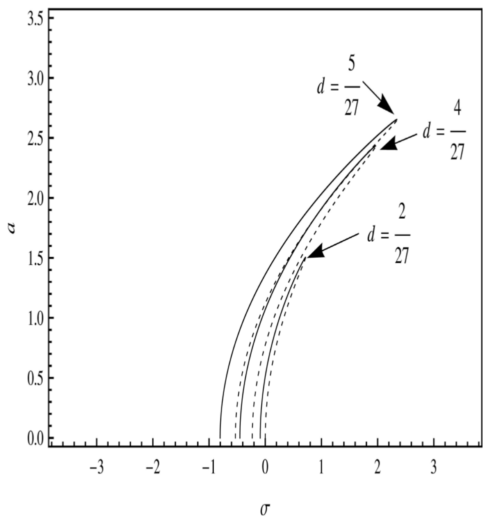

Figure 7. ***The frequency response curves for different values of*** d

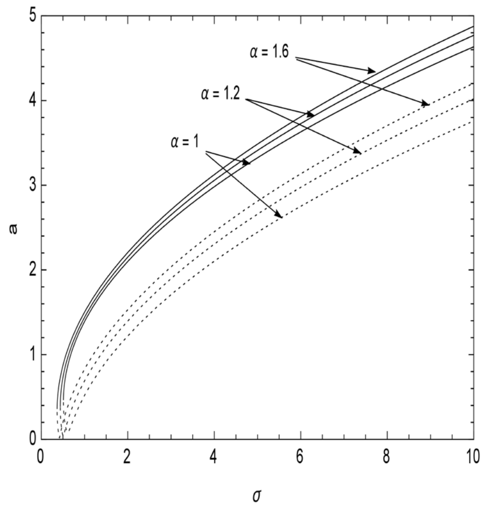

Figure 8. ***The frequency response curves for different values of*** α

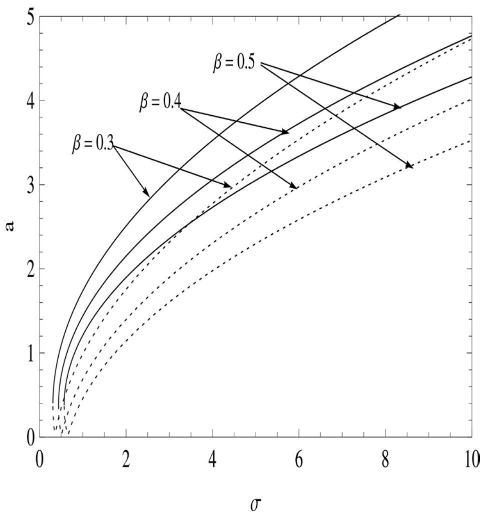

Figure 9. ***The frequency response curves for different values of*** β

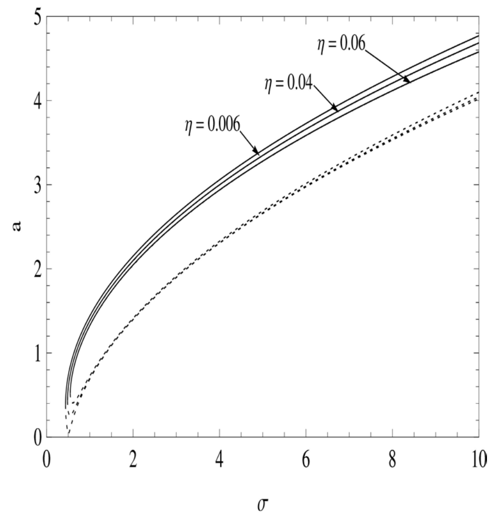

Figure 10. ***The frequency response curves for different values of*** η

From Fig.7 we observe that by increasing d, we have two solutions; one of them is stable and the other is unstable and the bend in the R.H.S.

Figures (8-14) represent the frequency response curves for the subharmonic solution of order 1/3 for certain values of the parameters $\alpha = 1.2, \beta = 0.4, \eta = 0.006, f = 5, \Sigma = 0.2, \zeta = 0.001, d = 4/27$ and $\gamma = 90$.

Fig.8 shows the variation of the amplitude of the steady state solutions for different values of α. It can be seen from this figure by decreasing α that the threshold becomes smaller and smaller and shifted to the left, the branches of the response curves diver from each other, the curves bent to the right of the σ axis and there exist two solutions; one of them is stable and the other is unstable.

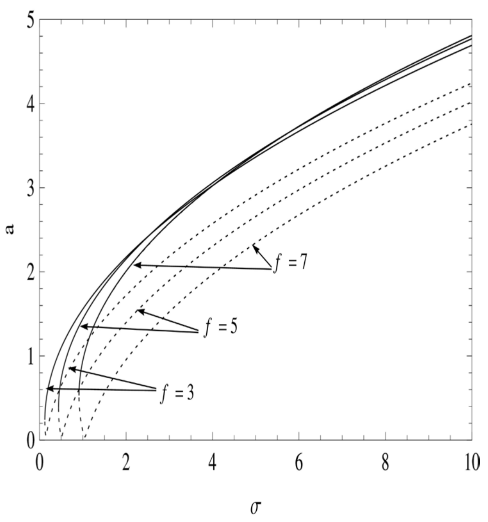

Figure 11. ***The frequency response curves for different values of*** f

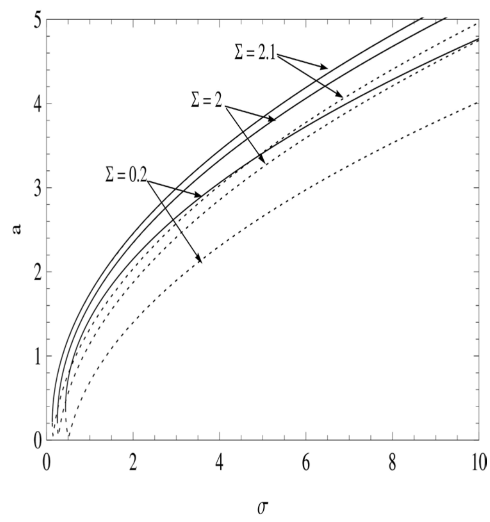

Figure 12. ***The frequency response curves for different values of*** Σ

From Fig.9, we note that by increasing β, the curves bent to the R.H.S, the threshold becomes smaller and smaller and we have two solutions, one of them is stable and the other is unstable.

From Fig.10 by increasing η, we have two solutions, one of them is stable and the other is unstable, the threshold becomes smaller and smaller and the inclination in the R.H.S.

Fig.11 shows that by increasing f the response curves are not strongly affected by small values and they are affected and shifted to the right for large values. Also, we have two solutions for a certain value of σ; one of them is stable and the other is unstable and the inclination in the R.H.S.

From Fig.12, we observe that by increasing Σ, the response curves are not strongly affected by small values and they are affected and shifted to the left for large values. Also, we have two solutions for a certain value of σ; one of them is stable and the other is unstable and the bend in the R.H.S.

Fig.13 shows that by increasing ζ, we obtain two solutions for a certain value of σ; one of them is stable and the other is unstable. The threshold becomes larger and larger; the branches does not change and the bend in the R.H.S.

From Fig.14, we note that by increasing d, the curves shifted and bent to the R.H.S, the threshold becomes larger and larger and we have two solutions; one of them is stable and the other is unstable.

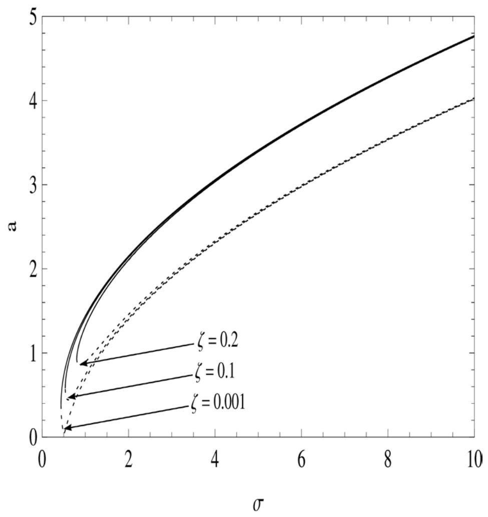

Figure 13. ***The frequency response curves for different values of*** ζ

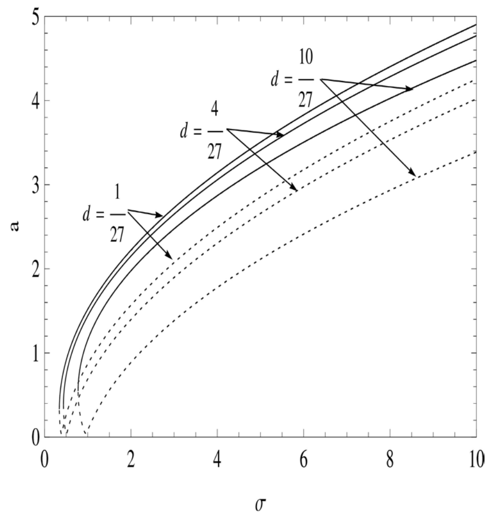

Figure 14. ***The frequency response curves for different values of*** d

6. Conclusion

In this work, we investigated a perturbation analysis for a weakly nonlinear second order differential equation based on tapping mode atomic force microscopy micro-cantilever dynamic behavior, using the concept of multiple scales. In addition, we concentrated on studying two other classes of periodic solutions, mainly subharmonic solutions of order $\frac{1}{n}$, $(n = 2, 3)$. Together with the modulation equations of the amplitude and phase, each type of solution included the frequency response equations, steady state solutions, peak amplitudes and their locations, and approximate analytical formulas. Furthermore, numerical solutions of the frequency response equations and the stability criteria were carried out to illustrate how the parameters impact the results. Finally, several figures were presented to show the validity of the results.

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