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# Logarithmic with-zero-one distribution for count data modeling with excess zeroes and ones

C. Satheesh Kumar and A. Riyaz

*Department Statistics, University of Kerala*

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**ABSTRACT**
Recently count data with excess zeroes and ones have received much attention in the literature due to its empirical needs and applications. Here we consider a new class of zero-one inflated version of the logarithmic-with-zero series distribution through its probability mass function. We derive the probability generating function, mean and variance of the proposed class of distribution and obtain expressions for its r-th raw moment and r-th factorial moments. The parameters of the distribution are estimated by the method of maximum likelihood and certain likelihood ratio test procedures are suggested. Further, the distribution has been fitted to certain real life data sets and thereby shown that the proposed model gives better fit to the data sets compared to existing models.

**KEYWORDS**
probability generating function; maximum likelyhood estimato; zero-one inflated distributions; factorial moments.

## 1. Introduction

Modelling of count data plays an important role in several areas of research such as manufacturing defect, public health, epidemiology, accident analysis and insurance etc. One of the commonly used models for modelling the count data is logarithmic series distribution (LSD) sometime called logarithmic distribution or log-series distribution.

In statistical literature several researchers concentrated their attention for developing distributions to handle zero-inflated data sets. Usually they construct the models through adding an extra parameter that specifically regulates the excess zero counts under predicted by the assumed standard distribution. Neyman (1939) and Feller (1943) introduced the concept of zero inflation towards addressing the problem of excess zeros. Zero inflated versions of the important discrete distributions were studied in the literature by several researchers. For reference see Lampert (1992), Cheung (2002), Neelon et al. (2010), Aryuyuen et al. (2014), Saengthong et al. (2015), Kumar and Riyaz (2013, 2014, 2015) etc.

In some practical problems count data with excess zeros and ones can be seen in many areas such as road safety, manufacturing defects, patient applications etc.

CONTACT C. Satheesh Kumar. Email: drcsatheeshkumar@gmail.com

Some examples of zero-one inflation can be the number of visits to a dentist for each patient per year for a sample of patients, the number of buying or not buying goods in shopping, the number of patients with a rare disease in different areas etc. For example, see Alshkaki (2016), Zhang et al. (2016), Liu et al. (2018) etc.

The logarithmic series distribution (LSD), due to Fisher et al. (1943 ), has been found applications in several areas of research such as biology, ecology, economics, inventory models, marine sciences etc. An important limitation of the LSD in certain practical situations is that it excludes the zero observation from its support. To overcome this difficulty the logarithmic-with-zeros distribution (LWZD) also known as the log-zero distribution was introduced by Williams (1947). Khatri (1961) studied the LWZD through the following probability mass function (pmf) in which $0 < \nu,\theta < 1$

$$f(x) = \begin{cases} \nu & \text{if } x = 0 \\ \dfrac{(1-\nu)\theta^x}{-xln(1-\theta)} & \text{if } x = 1, 2, 3, ... \end{cases} \tag{1}$$

In this paper we consider the zero-one inflation of the LWZD defined in (1). In section 2 we define logarithmic with zero-one inflated distribution through its pmf and obtain expression for its probability generating function (pgf). In section 3 we derive the expressions for its mean, variance, r-th raw moments and r-th factorial moments. In section 4, we describe the estimation of the parameters of the distribution by method of maximum likelihood and carried out certain test procedures. In section 5 we illustrate the usefulness of the model through fitting the distribution to certain real life data sets.

## 2. Definition and probability generating function

In this section first we present the definition of the proposed distribution and obtain its pgf.

### *Definition 2.1*

A non-negative integer valued random variable $X$ is said to follow the logarithmic with-zero-one distribution (LZOD) if its pmf $g(x) = P(X = x)$ is the following, in which $0 < \lambda_1, \lambda_2, \theta < 1$ such that $\lambda_1 + \lambda_1 < 1$ .

$$g(x; \lambda_1, \lambda_2, \lambda_2, \theta) = \begin{cases} \lambda_1 & \text{if } x = 0 \\ \lambda_2 - \dfrac{\theta}{ln(1-\theta)} & \text{if } x = 1, \\ \dfrac{(1-\lambda_1-\lambda_2)\theta^x}{-xln(1-\theta)} & \text{if } x = 2, 3, 4, ... \end{cases} \tag{2}$$

Clearly when $\lambda_2 = 0$ , the pmf given in (2) reduces to the pgf of the LWZD as given in (1); when $\lambda_1 = 0$ and $\lambda_2 = 0$ , the pmf (2) reduces to the pmf of the standard LSD introduced by Fisher et al. (1943). Now we derive the pgf of the LZOD through the following theorem.

**Theorem** 2.1 The pgf of the LZOD with pmf (2) is the following.

$$G(t) = (\lambda_1 + \lambda_2 t) - (\lambda_1 + \lambda_2)\frac{\theta t}{ln(1-\theta)} + (1 - \lambda_1 - \lambda_2)\frac{ln(1-\theta t)}{ln(1-\theta)} \tag{3}$$

**Proof**: By the definition of the pgf we have

$$\begin{aligned} G(t) &= \sum_{x=0}^{\infty} g(x)t^x \\ &= \lambda_1 + \left[\lambda_2 - \frac{\theta}{ln(1-\theta)}\right] t + \frac{(1-\lambda_1-\lambda_2)}{-ln(1-\theta)} \sum_{x=2}^{\infty} \frac{(\theta t)^x}{x} \\ &= \lambda_1 + \left[\lambda_2 - \frac{\theta}{ln(1-\theta)}\right] t + \frac{(1-\lambda_1-\lambda_2)}{-ln(1-\theta)} \left[\sum_{x=1}^{\infty} \frac{(\theta t)^x}{x} - \theta t\right], \end{aligned}$$

which gives (3)

**Remark 2.1** The pgf given in (3) has the following representation in terms of the Gauss hypergeometric function as given below.

$$G(t) = (\lambda_1 + \lambda_2 t) - (\lambda_1 + \lambda_2)\frac{t}{{}_2F_1(1;1;2;\theta)} + (1 - \lambda_1 - \lambda_2)t\frac{{}_2F_1(1;1;2;\theta t)}{{}_2F_1(1;1;2;\theta)} \tag{4}$$

**Remark 2.2** If we replace t by $e^{it}$ in (3) we get the characteristic function $\phi_x(t)$ of the LZOD as given below in which $t \in R$ and i= $\sqrt{-1}$ .

$$\phi_X(t) = (\lambda_1 + \lambda_2 e^{it}) - (\lambda_1 + \lambda_2)\frac{\theta e^{it}}{ln(1-\theta)} + (1 - \lambda_1 - \lambda_2)\frac{ln(1-\theta e^{it})}{ln(1-\theta)} \tag{5}$$

## 3. Properties

In this section we derive some other properties of the LZOD through the following results such as expression for its mean, variance, the r-th raw moment and the r-th factorial moment.

**Theorem 3.1** The mean and variance of the LZOD with pgf (3) are

$$mean = \lambda_2 - (\lambda_1 + \lambda_2)\frac{\theta}{ln(1-\theta)} - (1 - \lambda_1 - \lambda_2)\frac{\theta}{(1-\theta)ln(1-\theta)} \tag{6}$$

and

$$variance = \lambda_2 - \frac{\theta(1-\lambda_1-\lambda_2)}{(1-\theta)^2 ln(1-\theta)} - \frac{\theta(\lambda_1+\lambda_2)}{ln(1-\theta)} - \left[\lambda_2 - \frac{\theta(\lambda_1+\lambda_2)}{ln(1-\theta)} - \frac{\theta(1-\lambda_1+\lambda_2)}{(1-\theta)ln(1-\theta)}\right]^2 \quad (7)$$

Proof follows from the fact that

$$mean = G^{(1)}(1)$$

and

$$variance = G^{(2)}(1) + G^{(1)}(1) - \left[G^{(1)}(1)\right]^2$$

where

$$G^{(r)}(t) = \frac{d^r G(t)}{dt^r}/t = 1.$$

**Theorem 3.2**. For $r \geq 1$ the r-th raw moment $\mu_r^{'}$ of the LZOD with characteristic function (5) is

$$\mu_r^{'} = \lambda_2 - \left(\frac{\theta(\lambda_1+\lambda_2)}{ln(1-\theta)}\right) + \frac{(1-\lambda_1-\lambda_2)}{ln(1-\theta)}\Phi(\nu_2, 1-r, 1) \quad (8)$$

in which

$$\Phi(\nu_2, 1-r, 1) = \sum_{j=0}^{\infty} \lambda_2^j (1+j)^{r-1}$$

is the Lerch function, for any real $\lambda_2$ and $r \geq 1$

**Proof :**From (5) we have

$$\phi_X(t) = \sum_{r=1}^{\infty} \mu_r^{'} \frac{(it)^r}{r!} \quad (9)$$

$$= (\lambda_1 + \lambda_2 e^{it}) - (\lambda_1+\lambda_2)\frac{\theta}{ln(1-\theta)}e^{it} + (1-\lambda_1-\lambda_2)\frac{ln(1-\theta e^{it})}{ln(1-\theta)}. \quad (10)$$

On expanding the logarithmic function in (10), we get

$$\phi_X(t) = (\lambda_1 + \lambda_2 e^{it}) - (\lambda_1+\lambda_2)\frac{\theta}{ln(1-\theta)}e^{it} + \frac{(1-\lambda_1-\lambda_2)}{ln(1-\theta)}\sum_{j=0}^{\infty}\frac{(\theta e^{it})^{j+1}}{j+1}$$

$$= (\lambda_1 + \lambda_2 e^{it}) - (\lambda_1+\lambda_2)\frac{\theta}{ln(1-\theta)}e^{it} + \frac{(1-\lambda_1-\lambda_2)}{ln(1-\theta)}\sum_{j=0}^{\infty}\frac{\theta^{j+1}}{j+1}\sum_{r=0}^{\infty}(j+1)^r\frac{(it)^r}{r!}.$$

Now we expanding the exponential functions, we obtain

$$\phi_X(t) = \left(\lambda_1 + \lambda_2 \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\right) - \left(\frac{(\lambda_1+\lambda_2)\theta}{ln(1-\theta)} \sum_{r=0}^{\infty} \frac{(it)^r}{r!}\right) + \frac{(1-\lambda_1-\lambda_2)}{ln(1-\theta)} \sum_{r=0}^{\infty}\sum_{j=0}^{\infty} \theta^{j+1}(1+j)^{r-1}\frac{(it)^r}{r!} \tag{11}$$

On equating the coefficient of $\frac{(it)^r}{r!}$ on the right hand side expression of the (9) and (11), we get (8).

**Theorem 3.3** For $r \geq 1$, an expression for the r-th factorial moment $\mu_{[r]}$ of the LZOD is the following.

$$\mu_{[r]} = \begin{cases} \lambda_2 - \dfrac{(\lambda_1+\lambda_2)\theta}{ln(1-\theta)} + \dfrac{(1-\lambda_1-\lambda_2)\theta}{(1-\theta)ln(1-\theta)}, & \text{if } r = 1 \\ \dfrac{(1-\lambda_1-\lambda_2)}{ln(1-\theta)}(r-1)!(1-\theta)^{-r}\theta^r, & \text{if } r \geq 2, \end{cases} \tag{12}$$

Proof. By definition, the factorial moment generating function $F_X(t)$ of the LZOD with pgf (3) is

$$F_X(t) = \sum_{r=0}^{\infty} \mu_{[r]} \frac{t^x}{x!} = G(1+t)$$

$$= [\lambda_1 + \lambda_2(1+t)] - (\lambda_1+\lambda_2)\frac{\theta(1+t)}{ln(1-\theta)} + (1-\lambda_1-\lambda_2)\frac{ln[1-\theta(1+t)]}{ln(1-\theta)}$$

$$= [\lambda_1 + \lambda_2(1+t)] - (\lambda_1+\lambda_2)\frac{\theta(1+t)}{ln(1-\theta)} + \frac{(1-\lambda_1-\lambda_2)}{ln(1-\theta)} ln\left\{(1-\theta)[1-\frac{\theta}{1-\theta}t]\right\}$$

$$= [\lambda_1 + \lambda_2(1+t)] - (\lambda_1+\lambda_2)\frac{\theta(1+t)}{ln(1-\theta)} + \frac{(1-\lambda_1-\lambda_2)}{ln(1-\theta)}\left\{ln(1-\theta) + ln[1-\frac{\theta}{1-\theta}t]\right\} \tag{13}$$

.

On expanding the logarithmic term in (13) and equating the coefficient of $\frac{t^r}{r!}$ we obtain (12).

## 4. Estimation

In this section we discuss the estimation of the parameters $\lambda_1, \lambda_2$ and $\theta$ of the LZOD by the method of maximum likelihood estimation.

Let $\tau(x)$be the frequency of x events observed and let y be the highest value of the x observed. Then the likelihood function of the sample is

$$L = [f(0)]^{\tau(0)}[f(1)]^{\tau(1)} \prod_{x=2}^{y} [f(x)]^{\tau(x)} \tag{14}$$

where f(x) is the pmf of the LZOD as given in (2). Taking logarithm on both sides of (14) we get

$$logL = \tau(0)ln[f(0)] + \tau(1)ln[f(1)] + \sum_{x=2}^{y} \tau(x) \left[ln(1-\lambda_1-\lambda_2) + xln\theta - lnx + ln[ln(1-\theta)]\right] \tag{15}$$

Let $\hat{\lambda_1}, \hat{\lambda_2}$ and $\hat{\theta}$ denote the maximum likelihood estimators of the parameters $\lambda_1, \lambda_2$ and $\theta$ of the LZOD. On differentiating (15) partially with respect to the parameters $\lambda_1, \lambda_2$ and $\theta$ respectively and equating to zero, we get the following likelihood equations.

$$\tau(0)\lambda_1^{-1} - \sum_{x=2}^{\infty} \tau(x)(1-\lambda_1-\lambda_2)^{-1} = 0 \tag{16}$$

$$\frac{\tau(1)}{\lambda_2 - \dfrac{\theta}{ln(1-\theta)}} - \sum_{x=2}^{\infty} \tau(x)(1-\lambda_1-\lambda_2)^{-1} = 0 \tag{17}$$

and

$$\frac{\tau(1)}{\lambda_2 - \dfrac{\theta}{ln(1-\theta)}} \left[\frac{-1}{ln(1-\theta)} + \frac{\theta}{(1-\theta)[ln(1-\theta)]^2}\right] + \sum_{x=2}^{\infty} \tau(x) \left[\frac{x}{\theta} - \frac{1}{(1-\theta)ln(1-\theta)}\right] = 0 \tag{18}$$

When likelihood equation do not have a solution, the maximum of the likelihood function attained at the border of the domain of parameters. So we obtained the second order partial derivatives of $ln[f(x)]$ with respect to the parameters $\lambda_1, \lambda_2$ and $\theta$ and by using some mathematical software such as MATHLAB, MATHCAD, MATHEMATICA we observed that these equations give negative values for all $0 < \lambda_1, \lambda_2, \theta < 1$ such that $\lambda_1 + \lambda_2 < 1$ . Thus the density of the LZOD is log-cocave and hence the maximum likelihood estimators of the parameters $\lambda_1, \lambda_2$ and $\theta$ are unique (cf. Puig, 2003). Now, on solving the likelihood equations (16), (17) and (18) by using some mathematical software such as MATHLAB, MATHCAD and MATHEMATICA etc., one can obtain the maximum likelihood estimators of the parameters $\lambda_1, \lambda_2$ and $\theta$ of the LZOD.

## 5. Applications

In this section we illustrate all the procedures discussed in section 4 with the help of certain real life data sets. The first data is the frequency distribution of larval insect counts taken from Beall (1940), and the second data set is the distribution of European Corn-borer larvae Pyrausta Nubilalis in field corn from McGuire et al. (1957). We have fitted the LWZD, the zero-one inflated Piosson distribution (ZOIPD), the zero-one inflated negative binomial distribution (ZOINBD) and the LZOD to both the data sets and the results thus obtained along with the corresponding values of the expected frequencies, Chi-square statistic, degrees of freedom (d.f.), P, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and AICc. in respect of each of the models are presented in Tables 1 and 2. Based on the values of Chi-square statistic, d.f., P, AIC, BIC and AICc listed in tables 1 and 2, it can be observed that LZOD gives a better fit to both the data sets compared to the existing models – the LWZD, the ZOIPD and the ZOINBD.

**Table 1.** *The frequency distribution of larval insect counts taken from (Beall, 1940) and the expected frequencies computed using the LWZD, the ZOIPD, the ZOINBD and the LZOD distribution..*

| Count | Observed | LWZD | ZOPID | ZOINBD | LZOD |
|---|---|---|---|---|---|
| 0 | 43 | 64.80 | 45.33 | 53.28 | 43.20 |
| 1 | 35 | 39.47 | 41.71 | 39.93 | 35.91 |
| 2 | 17 | 10.06 | 19.63 | 10.45 | 16.01 |
| 3 | 11 | 3.42 | 7.13 | 6.37 | 10.01 |
| 4 | 5 | 1.31 | 5.77 | 5.10 | 5.06 |
| 5 | 4 | 0.53 | 0.35 | 2.37 | 3.88 |
| 6 | 1 | 0.23 | 0.06 | 1.45 | 3.10 |
| 7 | 2 | 0.10 | 0.001 | 0.88 | 2.56 |
| 8 | 2 | 0.08 | 0.02 | 0.18 | 0.27 |
| Total | 120 | 120 | 120 | 120 | 120 |
| Estimates of the parameters | | $\hat{\nu}$=0.51 | $\hat{p_0}$=0.01 | $\hat{\alpha}$=0.21 | $\hat{\lambda_1}$=0.36 |
| | | $\hat{\theta}$=0.54 | $\hat{p_1}$=0.99 | $\hat{\beta}$=0.19 | $\hat{\lambda_2}$=0.001 |
| | | | $\hat{\theta}$=0.98 | $\hat{\theta}$=0.61 | $\hat{\theta}$=0.96 |
| | | | | $\hat{k}=1$ | |
| $\chi^2$-value | | 14.24 | 29.17 | 11.69 | 0.25 |
| degrees of freedom | | 1 | 1 | 2 | 1 |
| P-value | | <0.0001 | 0.009 | 0.02 | 0.99 |
| AIC | | 196.09 | 426.68 | 390.34 | 181.12 |
| BIC | | 199.33 | 435.042 | 398.702 | 188.56 |
| AICc | | 196.15 | 426.89 | 390.518 | 181.18 |

**Table 2.** *Distribution of European Corn-borer larvae Pyrausta Nubilalis in field corn (McGuire et al., 1957) and the expected frequencies computed using the LWZD, the ZOIPD, the ZOINBD and the LZOD distribution.*

| Count | Observed | LWZD | ZOPID | ZOINBD | LZOD |
|---|---|---|---|---|---|
| 0 | 907 | 914.98 | 919.16 | 923.16 | 907.02 |
| 1 | 275 | 282.07 | 265.18 | 265.15 | 274.85 |
| 2 | 88 | 66.29 | 85.16 | 71.91 | 85.12 |
| 3 | 23 | 20.77 | 17.36 | 23.43 | 22.12 |
| 4 | 3 | 11.89 | 9.14 | 12.35 | 7.17 |
| Total | 1296 | 1296 | 1296 | 1296 | 1296 |
| Estimates of the parameters | | $\hat{\nu}$=0.71 | $\hat{p_0}$=0.35 | $\hat{\alpha}$=0.61 | $\hat{\lambda_1}$=0.70 |
| | | $\hat{\theta}$=0.47 | $\hat{p_1}$=0.78 | $\hat{\beta}$=0.01 | $\hat{\lambda_2}$=0.07 |
| | | | $\hat{\theta}$=0.60 | $\hat{\theta}$=0.71 | $\hat{\theta}$=0.999 |
| | | | | | $\hat{k}=1$ |
| $\chi^2$-value | | 14.24 | 6.56 | 11.34 | 2.56 |
| df | | 2 | 1 | 1 | 1 |
| P-value | | 0.0001 | 0.01 | 0.001 | 0.0634 |
| AIC | | 2217.46 | 2242.86 | 2604.90 | 2054.52 |
| BIC | | 2242.12 | 2265.528 | 2625.57 | 2077.188 |
| AICc | | 2217.46 | 2242.88 | 2604.93 | 2054.536 |

## 6. Testing of the hypothesis

For testing the hypothesis $H_0 : \lambda_2 = 0$ against the alternative hypothesis $H_1 : \lambda_2 \neq 0$, we construct the generalized likelihood ratio test (GLRT) and Rao's efficient score test (REST) as follows. In case of (GLRT), the test statistic is

$$-2log\alpha = 2[logL(\hat{\underline{\Omega}}; x) - logL(\hat{\underline{\Omega}}; x)] \tag{19}$$

where $\hat{\underline{\Omega}}$ is the maximum likelihood estimator of $\underline{\Omega} = (\lambda_1, \lambda_2, \theta)$ with no restrictions, and $\hat{\underline{\Omega}}$ is the maximum likelihood estimator of $\underline{\Omega}$ when $\lambda_2 = 0$ . The test statistic $-2log\alpha$ given in (19) is asymptotically distributed as Chi-square with one degree of freedom. For details see Rao, (1973). We have computed the values of $logL(\hat{\underline{\Omega}}; x)$ , and $logL(\hat{\underline{\Omega}^*}; x)$the test statistic for the LZOD with $\lambda_2 = 0$ in case of the first and the second data sets are given in the Table 3.

**Table 3.** *The computed the values of $logL(\hat{\underline{\Omega}}; x)$ $logL(\hat{\underline{\Omega}^*}; x)$ and the generalized likelihood ratio test statistic under $H_0$*

| | $logL(\hat{\underline{\Omega}^*}; x)$ | $logL(\hat{\underline{\Omega}}; x)$ | Test statistics |
|---|---|---|---|
| Data set 1 | -207.038 | -201.171 | 11.734 |
| Data set 2 | -1185.001 | -1138.991 | 92.021 |

In case of (REST), the following test statistic can be used.

$$S = T^1 \phi^{-1} T, \tag{20}$$

where $T^{'} = \left( \frac{1}{\sqrt{n}} \frac{\partial lnL}{\partial \lambda_1}, \frac{1}{\sqrt{n}} \frac{\partial lnL}{\partial \lambda_2}, \frac{1}{\sqrt{n}} \frac{\partial lnL}{\partial \theta}, \right)$ and $\phi$ is the Fisher information matrix. The test statistic given in (20) follows Chi-square distribution with one degree of freedom (see Rao, 1973). We have also computed the values of based on (20) for the LZOD in the case of first data set as $S_1$ and for the LZOD in the case of second data set $S_2$ as given below.

$$S_1 = (4.904\ 4.678\ 10.97) \begin{bmatrix} 1,626 & -0.433 & -0.327 \\ -0.433 & 0.038 & 0.291 \\ -0.327 & 0.291 & 0.035 \end{bmatrix} \begin{bmatrix} 4.904 \\ 4.678 \\ 10.97 \end{bmatrix}$$

$$=19.093$$

$$S_2 = (4.904\ 4.678\ 10.97) \begin{bmatrix} 0.201 & -0.819 & -0.146 \\ -0.189 & 0.486 & -0.26 \\ -0.146 & -0.26 & 1,741 \end{bmatrix} \begin{bmatrix} 5.014 \\ 0.011 \\ 0.134 \end{bmatrix}$$

$$=5.046$$

Since the critical value for the test at 5 % level of significance and one degree of freedom is 3.84, the null hypothesis that $H_0; \lambda_2 = 0$ is rejected in both the above cases in respect of GLRT and REST.

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