Expected Returns and the Global minimum-variance Portfolio in the Zero-beta CAPM

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Abstract: We show that expected returns in the Black capital asset pricing model are driven only by the market portfolio and the global minimum-variance portfolio (GMVP). Expected return of a portfolio is a linear combination of expected market and GMVP returns, less a cost which is the present value of cash flows under the well-known Gordon Growth Model, where the growth rate is the expected return of the GMVP, and the discount rate is the expected market return.

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1. Introduction

The capital asset pricing model (CAPM) is a simple but intellectual formula that shows how market risk and expected return are related in a rational market. Under the Sharpe (1964) and Lintner (1965) CAPM, the expected return ER_i on any asset i is given by the equation:

$$ER_i = Rf + \beta_i E[R_m - R_f], \ \beta_i = \frac{\mathbf{Cov}(R_i, R_m)}{\mathbf{Var}\,R_m}$$

where R_f represents the riskless interest rate, and $E[R_m - R_f]$ denotes the market risk premium. Beta (β_i) is the market risk of asset i; it is the market risk–adjusted covariance of the return R_i on asset i, and the return R_m of the mean–variance efficient market portfolio. Hence, the Sharpe–Lintner CAPM is a one-factor model driven by the market portfolio, with beta being the sensitivity to the market (i.e, systematic market risk).

The Sharpe–Lintner CAPM builds on the model of portfolio choice originally de- veloped by Markowitz (1952, 1959). Black (1972) extends the CAPM by relaxing the assumption that investors can borrow and lend at the riskless rate and develops a version of the model where all assets are

risky, provided there is a zero–beta asset that proxies for the riskless asset. This zero-beta asset is uncorrelated to the market portfolio. The Black CAPM gives the expected return on an asset or portfolio i (with return R_i) in terms of the expected return on the zero–beta asset zm (with return R_{zm}), which is uniquely determined from any mean–variance efficient market portfolio m (with return R_m). Consequently, the expected return on any portfolio i is given as:

$$E[R_i - R_{zm}] = \beta i E[R_m - R_{zm}] \qquad (1.1)$$

$$\beta_i = \frac{\mathbf{Cov}(R_i, R_m)}{\mathbf{Var}(R_m)}$$

Let the expected return of asset "a" be $\mu_a = ER_a$, where $a \in \{i, g, m, z, mz, zm\}$. From equation (1.1), we get

$$E[R_i] = \tilde{\beta}_i E[R_{zm}] + \beta_i E[R_m], \qquad (1.3)$$

where $\tilde{\beta}_i \equiv 1 - \beta_i$ is the sensitivity to the zero-beta asset.

Academics and practitioners have collectively come to rely on the CAPM for de- termining the discount rate for valuing a firm, for valuing investments within a firm, for setting executive compensation, and for benchmarking fund managers (e.g., Bodie, Kane, and Marcus, 2005; Dempsey, 2013; Fama and French, 1992; Ferguson and Shock- ley, 2003; Jagannathan and Wang, 1996; Kandel and Stambaugh, 1995; Mehrling, 2007; Savov, 2011; Ukhov, 2006). Testing asset pricing models continue to be a major research topic in finance and while multi-factor formulas have emerged, the Sharpe-Lintner CAPM remains the foundational building block for these new models (see Carhart, 1997; Chiah, Chai, Zhong, and Li, 2016; Zhou, 1993; Subrahmanyam, 2010; Fama and French, 2015a, 2015b, 2016, 2017; Kubota and Takehara, 2018).

The Black CAPM receives much less attention in the literature. Yet, allowing for the absence of a risk-free asset is one of the most important extensions of the original Sharpe–Lintner model. To estimate and assess this version of the CAPM, Chou (2000) recommends a simple Wald test approach while Beaulieu, Dufour, Khalaf (2013) propose a simulation-based procedure as the zero-beta asset is unobservable which leads to more empirical difficulties compared to the riskless interest rate based approach.

Buckley et al. (2013) show that although the zero-beta asset is uncorrelated to the market portfolio, it moves inversely with market risk and becomes extremely volatile when market returns are close to the return of the global minimum-variance portfolio (GMVP). Moreover, they show

that the zero–beta asset uniquely determines the market portfolio, and conversely, the market portfolio uniquely determines the zero-beta portfolio on the mean-variance efficient frontier. Therefore, we may replace the expected return of the zero-beta asset by a non-linear function of the expected market return and the expected return of the GMVP.

As a direct consequence, we illustrate that the uncorrelated but risky zero-beta asset is not required in explaining expected returns as it becomes redundant in the presence of the market and global minimum-variance portfolios. This study contributes to the literature by showing that the expected return of a portfolio is a linear combination of the expected market return and the expected GMVP return less a cost/penalty. This cost is the present value of cash flows under the well-known Gordon Growth Model, where the growth rate is the expected return of the GMVP, and the discount rate is the expected market return. Unlike the zero-beta asset, the GMVP is positively correlated with the market portfolio, but this correlation vanishes whenever the market becomes extremely volatile. To the best of our knowledge, this is the first study to link the Black CAPM to Gordon's well-known infinite growth model.

2. The Black mean–variance frontier

In this section, we closely follow Buckley et al. (2013) by briefly reviewing the construc- tion and properties of the Black (1972) mean-variance frontier. Recall that there is no riskless asset in the Black CAPM framework. Instead, it is assumed that a zero–beta asset exists which is uncorrelated with the efficient market portfolio; it acts as proxy for the riskless asset.

Consider n assets, with returns $R_1, R_2, \cdots, R_n$, return vector $\mathbf{R} = (R_1, R_2, \cdots, R_n)'$ and expected return $\mu = \mathbf{ER} = (\mathbf{E}R_1, \mathbf{E}R_2, \cdot, \mathbf{E}R_n)'$. Black (1972) assumes that no linear combination of these risky assets has a variance of zero. Furthermore, the covariance matrix of returns, denoted by $\Omega = \mathbf{Cov}(\mathbf{R}, \mathbf{R}')$, is assumed to be non–singular, and hence, invertible. Let R_m be the return on a portfolio of assets built from these n assets, with weight vector $\mathbf{w}_m = (w_1, w_2, \cdots, w_n)'$. Since there are no risk–free assets, the portfolio m is fully invested in the assets $1, 2, \cdots, n$, with $R_m = \mathbf{wR}'$, $\mu_m = \mathbf{w}\mu'$, $\mathbf{wI}' = 1$ and $\mathbf{I}$ is a column of ones. The mean–variance frontier is the locus of (σ_m, μ_m), where

$$\sigma_m^2 = \min_w \mathbf{w}'\Omega\mathbf{w},$$

$$\text{subject to } \mu_m = \mathbf{w}\mu', \ \mathbf{wI}' = 1. \tag{2.1}$$

Thus, for a given mean return μ_m, the minimum variance of the return on the portfolio is σ_m^2. One can easily show that the weight vector $\mathbf{w}_m$ is a

linear function of the expected return μ_m of the portfolio (cf Campbell, Lo, and MacKinlay (1997))

$$\mathbf{w}_m = \mathbf{g} + \mathbf{h}\,\mu_m, \tag{2.2}$$

where the constant vectors $\mathbf{g}$ and $\mathbf{h}$ are built from the (inverse) covariance matrix Ω, the mean return vector μ and $\mathbf{I}$ as follows:

$$\mathbf{g} = \frac{1}{D}[B(\Omega^{-1}\mathbf{I}) - A(\Omega^{-1}\mu)], \quad \mathbf{h} = \frac{1}{D}[C(\Omega^{-1}\mu) - A(\Omega^{-1}\mathbf{I})].$$

The parameters A, B, C, and D are constants constructed from the inverse covariance matrix of the underlying base portfolio from which all portfolios are constructed and given explicitly as

$$A = \mu^T\Omega^{-1}\mathbf{I}, \quad B = \mu^T\Omega^{-1}\mu, \quad C = \mathbf{I}^T\Omega^{-1}\mathbf{I}, \quad D = BC - A^2. \tag{2.3}$$

We now give an explicit formula for the minimum variance conditioned on the expected return on the portfolio. The reader is directed to Campbell et al. (1997) for a proof of the following proposition.

Proposition 1. *The mean–variance frontier is the locus of* (σ_m, μ_m)*, where*

$$\sigma_m^2 = \frac{C}{D}\left(\mu_m - \frac{A}{C}\right)^2 + \frac{1}{C}. \tag{2.4}$$

Moreover, the covariance of returns R_m and R_q, where m and q are mean–variance portfolios, is

$$\mathbf{Cov}\,(R_m, R_q) = \frac{C}{D}\left(\mu_m - \frac{A}{C}\right)\left(\mu_q - \frac{A}{C}\right) + \frac{1}{C}. \tag{2.5}$$

2.1. The global minimum-variance portfolio

It follows immediately from Proposition 1, that σ_m^2, the variance of the market portfolio, has a global minimum of $\frac{1}{C}$ when the market return is μ_m = $\frac{A}{C}$. This unique portfolio is called the *global minimum-variance portfolio* (GMVP). The subscript "*g*" is used to reference GMVP. The global minimum-variance portfolio g, with return R_g, has mean, variance and weight vector, respectively given by

$$\mu_g = \frac{A}{C}, \quad \sigma_g^2 = \mathbf{Var}\, R_g = \frac{1}{C}, \quad \mathbf{w}_g = \frac{1}{C}\Omega^{-1}\,\mathbf{I} = \Omega^{-1}\mathbf{I}\sigma_g^2.$$

Moreover, the covariance between R_g and R_m, the return on any other mean–variance portfolio (MVP), is equal to the global minimum-variance $\frac{1}{C}$. It is easy to compute the correlation between the market and global minimum-variance portfolios, which is given in terms of market volatility as follows:

$$\rho_{mg} = \frac{1}{\sqrt{C\sigma_m^2}}. \tag{2.6}$$

Note that $\rho_{mg} \rightarrow 0$ when market volatility is very high. Clearly, the correlation be- tween the market portfolio and GMVP is positive and inversely proportional to market volatility. Furthermore, it decreases to zero at a rapid rate as market volatility gets in- creasingly larger. From Proposition 1, note that market volatility increases monotonically in either expected market or GMVP return since

$$\sigma_m^2 = \frac{C}{D}\left(\mu_m - \frac{A}{C}\right)^2 + \frac{1}{C} = \frac{C}{D}(\mu_m - \mu_g)^2 + \mu_g^2.$$

Therefore, the correlation coefficient also decreases monotonically (to zero) whenever either market return or return of the GMVP increases.

2.2. *The zero–beta asset*

We now introduce the zero–beta asset, *zm*, for each mean–variance market portfolio *m*. It is the MVP with return R_{zm}, which has minimum variance and zero covariance with the MVP portfolio having return R_m. Thus $\rho_{zm,m} =$ 0. This portfolio is unique. Unlike the risk–free asset, the zero–beta asset has mean and variance that are tied to the mean and variance of the market return R_m. Buckley et al. (2013) noted this fact in the following proposition.

Proposition 2. *Let R_m be the return of a MVP, with expected return μ_m, and volatility σ_m. Provided m is not the global minimum-variance portfolio GMVP, there exists a unique zero–beta portfolio zm, with return R_{zm} that is a MVP, with expected return μ_{zm} and volatility σ_{zm}, given respectively, by*

$$\mu_{zm} = \frac{A}{C} - \frac{\dfrac{D}{C^2}}{\mu_m - \dfrac{A}{C}}. \tag{2.7}$$

$$\sigma_{zm}^2 = \frac{1}{C^2}\left(\frac{1}{\sigma_m^2 - \frac{1}{C}}\right) + \frac{1}{C}. \tag{2.8}$$

It follows easily from equation (2.8) that $\sigma_{zm}^2 - \sigma_g^2 = \dfrac{1}{C^2}\dfrac{1}{\left(\sigma_p^2 - \sigma_g^2\right)}.$

Therefore, the ex-cess volatility of the zero-beta asset relative to the global minimum-variance portfolio is inversely proportional to the the excess volatility of the market portfolio relative to the global minimum-variance portfolio. In other words, using the GMVP as benchmark, the excess volatility of the zero-beta and market portfolios covary negatively.

For convenience, we link the returns and variances of the GMVP and zero–beta asset, as follows.

Corollary 1. *Let* $\mu_g = \dfrac{A}{C}$, *be the expected return and* **Var**$R_g = \dfrac{1}{C}$, *the variance of the GMVP. Then* ***provided*** $\mu_m \neq \mu_g$, *the expected return and variance of the unique zero–beta asset zm, with return* R_{zm}, *are respectively given by:*

$$\mu_{zm} = \mu_g - \frac{\dfrac{D}{C^2}}{\mu_m - \mu_g}$$

$$\mathbf{Var}\,R_m = \sigma_m^2 = \frac{C}{D}(\mu_m - \mu_g)^2 + \mathbf{Var}\,R_g$$

$$\mathbf{Var}\,R_{zm} = \sigma_{zm}^2 = \frac{1}{C^2}\left(\frac{1}{\mathbf{Var}R_m - \mathbf{Var}R_g}\right) + \mathbf{Var}R_g = \frac{\sigma_m^2 \sigma_g^2}{\sigma_m^2 - \sigma_g^2}$$

3. Expected return in terms of the GMVP

We now present the main result that the expected return on an asset depends only on the returns of the market and global minimum-variance portfolios.

Theorem 1. *Let* R_i *be the return on any asset i, with expected return* μ_i. *Let* μ_m *be the expected return on the market portfolio,* R_m *and* μ_g, *the expected return on the global minimum-variance portfolio, GMVP. Then*

$$\mu_i = \beta_i \mu_m + (1 - \beta_i)\mu_g - \frac{K_i}{\mu_m - \mu_g}, \tag{3.1}$$

where $\beta_i = \dfrac{\mathbf{Cov}(R_i, R_m)}{\mathbf{Var}(R_m)}$ *and* $K_i = \dfrac{(1 - \beta_i)D}{C^2}$ *is the initial cash flow in the Gordon Growth Model with growth rate* μ_g *and discount rate* μ_m.

Proof. From Equation (1.1), $\mathbf{E}[R_i] = \tilde{\beta}_i \mathbf{E}[R_{zm}] + \beta_i \mathbf{E}[R_m]$, *where* $\tilde{\beta}_i \equiv 1 - \beta_i$ is the sensitivity to the zero-beta asset. Thus $\mu_i = \beta_i \mu_m + \tilde{\beta}_i \mu_{zm}$. From Proposition 2 and Corollary 1, $\mu_{zm} = \dfrac{A}{C} - \dfrac{\dfrac{D}{C^2}}{\mu_m - \dfrac{A}{C}} = \mu_g - \dfrac{\dfrac{D}{C^2}}{\mu_m - \mu_g}$. Substituting μ_{zm} into the former equation

$$\mu_i = \beta_i \mu_m + \tilde{\beta}_i \mu_g - \frac{\dfrac{\tilde{\beta}_{im} D}{C^2}}{\mu_m - \mu_g} = \beta_i \mu_m + \tilde{\beta}_i \mu_g - \frac{K_i}{\mu_m - \mu_g},$$

where $K_i = \dfrac{(1 - \beta_i)D}{C^2}$ is the initial cash flow in Gordon's infinite growth model that has a growth rate of μ_g and discount rate μ_m.

Observe from equation (3.1) that the expected return is a linear combination of ex-pected market return and expected GMVP return, less a **cost** which depends on both expected market and GMVP returns. The cost is the present value of cash flows under the well-known Gordon Growth Model, where the initial cash flow is $K_i = \dfrac{(1 - \beta_i)D}{C^2}$, the growth rate is the expected return of the GMVP (denoted μ_g), while the discount rate is the expected market return (denoted μ_m). This representation clearly shows that under the Black CAPM the expected return of a portfolio has a linear and non-linear dependence on only the expected market and global minimum-variance portfolios. Consequently, the zero-beta asset is redundant.

4. Conclusion

Expected return of a portfolio in the Black CAPM is driven only by the market portfolio and the global minimum-variance portfolio (GMVP).

Consequently, the uncorrelated but risky zero-beta asset becomes redundant in explaining expected returns. In addition, we find that the expected return is a linear combination of expected market and GMVP returns minus a cost determined by the famous Gordon Growth Model. This cost is the present value of cash flows under the infinite growth paradigm, where the growth rate is the expected return of the GMVP and the discount rate is the expected market return. Unlike the zero-beta asset, the GMVP and market are positively correlated but decreases rapidly to zero whenever the market becomes extremely volatile.

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