Journal of Quantitative Finance and Economics. **2020**, 2, 2

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The Intrinsic Price of Jumps Associated with Hedging Strategies

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Received: 12 July 2020; Revised: 23 July 2020;
Accepted: 8 September 2020; Publication: 07 October 2020

Abstract: We propose a problem of intrinsic price of jumps associated with hedging strategies in an incomplete market where the stock price follows Merton jump diffusion model. Then we give a representation formula of this intrinsic price in view of options, and derive that there is a minimizing intrinsic price of jumps associated with hedging strategies in this incomplete market.

Keywords: intrinsic price; incomplete market; jump diffusion model; minimizing intrinsic price

JEL classification: C22; G13

Introduction

It is well known that the hedging strategy of the classical Black-Scholes model (1973) is made of one share and a number of options ($1/\Delta$). J. C. Hull (1997) adopted the hedging strategy containing one option and a number of shares (Δ). In these cases they all assumed that the option price is a function of the stock price and time to expiration, i.e. the option price is $F = F(S, t)$.

The so-called incomplete market in this paper means exactly the stock price follows Merton jump diffusion model (1973). We consider the intrinsic price of jumps associated with hedging strategy (M, N), where N is the number of shares and M is the number of options. The hedging strategy (M, N) in this paper is nonstandard, i.e. the hedging proportion $\frac{N}{M} \neq \Delta$ and the option price F denoted by $F = F(S, t, N)$ also depends on the number of shares.

The rest of the paper is organized as follows. In section 2 we arrive at the intrinsic price of jumps associated with hedging strategies. Then we derive and obtain a minimum intrinsic price p* and the corresponding hedging strategy N^* in section 3.

2. The Intrinsic Price of Jumps Associated with Hedging Strategies

Anatoly B. Schmidt (2003) studied the option price $F_1' = F_1'(S,t,N)$ only for the stock price following a geometric Brownian motion

$$dS_t = \mu S_t\, dt + S_t\, \sigma\, dB_t \tag{2.1}$$

associated with the hedging strategy (M_1, N). He obtained

$$F'_1(S,t,N) = F_1(S,t)N^{a_1} \tag{2.2}$$

where $a_1 = \frac{S}{F_1}\frac{\partial F_1}{\partial S}$, F_1 is the solution of BS equation when $N = 1$.

In this paper we first give out the option price $F'_2 = F'_2(S,t,N)$ for the stock price following a Merton jump-diffusion model. That is the stock price follows

$$dS_t = S_t\,(\mu - \lambda k)dt + S_t\,\sigma dB_t + S_t\,\rho dJ_t \tag{2.3}$$

where B_t is a standard Brownian motion and J_t is a Possion process with jumping frequency $\lambda > 0$. μ is the instantaneous expected return on the stock price S_t and σ is the instantaneous variance of the return when the Possion event does not occur. ρ is the proportion of the total volatility due to jumps and ρ_i are i.i.d random variables with the expectation k where i = 1, 2,…, J_t. Here B_t, J_t and ρ_i are mutually independent. For convenience the subscript t is omitted.

Lemma 2.1 Assume the stock price satisfies (2.3) and the option price F'_2 has the form $F'_2\,(S, t, N) = F_2\,(S, t)\, Z_2\,(N)$ where $Z_2\,(1) = 1$. Then we have the following representation

$$F_2'(S,t,N) = F_2(S,t)N^{a_2} \tag{2.4}$$

where F_2 is consistent with the case when $N = 1$ in Merton jump diffusion model and $a_2 = \frac{S}{F_2}\frac{\partial F_2}{\partial S}$.

Proof: We first discrete the equation (2.3) into

$$\Delta S = S(\mu - \lambda \mathrm{k})\Delta t + S\sigma\Delta B + S\rho\Delta J + o(\cdot) \tag{2.5}$$

Then we have the following equation

$$\Delta F_2' = \frac{\partial F_2'}{\partial S}\Delta S + \frac{\partial F_2'}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F_2'}{\partial S^2}\Delta S^2 + \frac{\partial F_2'}{\partial N}\Delta N + o(\cdot) = \frac{\partial F_2'}{\partial S}S(\mu - \lambda k)\Delta t + \frac{\partial F_2'}{\partial S}S\sigma\Delta B + \frac{\partial F_2'}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F_2'}{\partial S^2}S^2\sigma^2\Delta t + \frac{\partial F_2'}{\partial S}S\rho\Delta J + \frac{1}{2}\frac{\partial^2 F_2'}{\partial S^2}S^2\rho^2\lambda\Delta t + \frac{\partial F_2'}{\partial N}\Delta N + o(\cdot). \tag{2.6}$$

Consider the portfolio (M_2, N) with N shares and M_2 options whose price is $F_2' = F_2'(S,t,N)$. The value of the portfolio can be described as $V = M_2F_2' + NS$ and satisfies

$$\Delta V = M_2\Delta F_2' + N\Delta S + S\Delta N = M_2\left[\frac{\partial F_2'}{\partial S}S(\mu - \lambda k) + \frac{1}{2}\frac{\partial^2 F_2'}{\partial S^2}S^2\sigma^2 + \frac{1}{2}\frac{\partial^2 F_2'}{\partial S^2}S^2k^2\lambda\right]\Delta t + NS(\mu - \lambda k)\Delta t + \left(M_2\frac{\partial F_2'}{\partial S}S\sigma + NS\sigma\right)\Delta B + \left(M_2\frac{\partial F_2'}{\partial S}Sk + NSk\right)\Delta J + \left(M_2\frac{\partial F_2'}{\partial N} + S\right)\Delta N + o(\cdot). \tag{2.7}$$

Substitute the variable ρ with its expectation k in order to eliminate the uncertainty, and by using the arbitrage-free condition, the value of this portfolio can be naturally described as

$$\Delta V = rV\Delta t = r(M_2 F'_2 + NS)\Delta t. \tag{2.8}$$

Based on (2.7) and (2.8) and we get the following three equations

$$M_2\frac{\partial F_2'}{\partial S} + N = 0, \tag{2.9}$$

$$M_2\frac{\partial F_2'}{\partial N} + S = 0, \tag{2.10}$$

$$M_2\left[\frac{\partial F_2'}{\partial S}S(\mu - \lambda k) + \frac{1}{2}\frac{\partial^2 F_2'}{\partial S^2}S^2\sigma^2 + \frac{1}{2}\frac{\partial^2 F_2'}{\partial S^2}S^2k^2\lambda\right] + NS(\mu - \lambda k) - r(NS + M_2F_2') = 0. \tag{2.11}$$

Since $F_2'(S,t,N) = F_2(S,t)Z_2(N)$ and Z_2 (1) = 1, we have

$$M_2 = \frac{-N}{Z_2\frac{\partial F_2}{\partial S}}, \tag{2.12}$$

$$\frac{dZ_2}{dN} = \frac{S}{F_2}\frac{\partial F_2}{\partial S}\frac{Z_2}{N}, \tag{2.13}$$

$$\frac{\partial F_2}{\partial t} + \frac{1}{2}\frac{\partial^2 F_2}{\partial S^2}S^2\sigma^2 + \frac{1}{2}\frac{\partial^2 F_2}{\partial S^2}S^2k^2\lambda + rS\frac{\partial F_2}{\partial S} - rF_2 = 0. \tag{2.14}$$

From the equation (2.14) we know F_2 is consistent with the case when $N = 1$ in Merton jump diffusion model. Combining (2.13) with the condition

$Z_2(1) = 1$ we easily get $Z_2(N) = N^{a_2}$, where $a_2 = \frac{S}{F_2}\frac{\partial F_2}{\partial S}$. This ends the proof of Lemma 2.1.

Defination 2.2: The intrinsic price P of jumps associated with hedging strategies in view of options in an incomplete market is defined by $P(k, \lambda, N) \triangleq F_2' - F_1'$.

Theorem 2.3: The intrinsic price $P(k, \lambda, N)$ in the incomplete market is described as

$$P(k, \lambda, N) = F_2(N^{a_2} - N^{a_1}) + \Delta F N^{a_1} \tag{2.15}$$

where $\Delta F = F_2 - F_1$.

Proof: It is not hard to show from Lemma 2.1 that Theorem 2.1 is tenable.

3. A Minimizing Intrinsic Price of Jumps Associated With (*M*, *N*)

Without loss of generality we consider the European call option with a strike E and time to expiration $\tau = T - t$. From (2.1) we known the option price F_1 can be described as

$$F_1(S, t) = S\Phi(d_+^1) - Ee^{-r\tau}\Phi(d_-^1) \tag{3.1}$$

where Φ is the standard normal cumulative distribution and

$$d_+^1 = \frac{ln\frac{S}{E} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}; d_-^1 = d_+^1 - \sigma\sqrt{\tau}. \tag{3.2}$$

The equation (2.14) can be obtained if we substitute σ^2 with $\sigma^2 + k^2\lambda$, in the classical BS equation. So the solution of (2.14) is

$$F_2(S, t) = S\Phi(d_+^2) - Ee^{-r\tau}\Phi(d_-^2) \tag{3.3}$$

where

$$d_+^2 = \frac{ln\frac{S}{E} + (r + \frac{1}{2}\sigma^2 + \frac{1}{2}k^2\lambda)\tau}{\sqrt{\sigma^2 + k^2\lambda}\sqrt{\tau}}; d_-^2 = d_+^1 - \sqrt{\sigma^2 + k^2\lambda}\sqrt{\tau}. \tag{3.4}$$

Theorem 3.1: If it holds $d^1_- < 0$and $d^1_+ > 0$, there exists $N^* > 1$ so that the intrinsic price $P(k, \lambda, N)$ achieves a minimum $P^* > 0$ at the point N^*.

Proof: Since the option price will go up when the volatility increases, $\Delta F = F_2 - F_1$ is positive. According to Lagrange Mean Value Theorem we have

$$N^{a_2} - N^{a_1} = N^{\xi} lnN(a_2 - a_1) \tag{3.5}$$

where ξ is located between a_1 and a_2. From Theorem 2.3 we have

$$P(k,\lambda,N) = F_2(a_2 - a_1)N^{\xi} lnN + \Delta F N^{a_1}. \tag{3.6}$$

Now we have the first-order partial derivative

$$\frac{\partial P}{\partial N} = F_2(a_2 - a_1)N^{\xi-1}(\xi lnN + 1) + \Delta F a_1 N^{a_1-1}. \tag{3.7}$$

A key point is to compare the size of a_1 with a_2. Noticing $\frac{\partial F_1}{\partial S} = \Phi(d_+^1)$ and $\frac{\partial F_2}{\partial S} = \Phi(d_+^2)$ we have

$$a_1 = \frac{S}{F_1}\frac{\partial F_1}{\partial S} = \frac{S\Phi(d_+^1)}{S\Phi(d_+^1) - Ee^{-r\tau}\Phi(d_-^1)} = \frac{1}{1 - \frac{Ee^{-r\tau}}{S}\frac{\Phi(d_-^1)}{\Phi(d_+^1)}}, \tag{3.8}$$

$$a_2 = \frac{S}{F_2}\frac{\partial F_2}{\partial S} = \frac{S\Phi(d_+^2)}{S\Phi(d_+^2) - Ee^{-r\tau}\Phi(d_-^2)} = \frac{1}{1 - \frac{Ee^{-r\tau}}{S}\frac{\Phi(d_-^2)}{\Phi(d_+^2)}}. \tag{3.9}$$

From (3.2) and the condition $d^1_- < 0$ and $d^1_+ > 0$ we can get the first-order partial derivative

$$\frac{\partial d_+^1}{\partial \sigma^2} = \frac{-d_-^1}{2\sigma^2} > 0, \tag{3.10}$$

$$\frac{\partial d_-^1}{\partial \sigma^2} = \frac{-d_+^1}{2\sigma^2} < 0. \tag{3.11}$$

So it holds $d^1_+ < d^2_+$ and $d^1_- > d^2_-$ and then $0 < a_2 < a_1$. Now we rewrite (3.7) as

$$\frac{\partial P}{\partial N} = N^{a_1-1}\left[F_2(a_2 - a_1)N^{\xi-a_1}(\xi lnN + 1) + \Delta F a_1\right]. \tag{3.12}$$

It is easy to observe $\frac{\partial P}{\partial N}$ is negative when $N = 1$ and positive when N tends to infinity. Then there exists $N^* > 1$ satisfying

$$\frac{\partial P}{\partial N}|_{N=N^*} = 0 \tag{3.13}$$

and the intrinsic price $P(k, \lambda, N)$ obtains a minimum P^* at the point N^*. In order to compute the value of N^*, we obtain from (2.15)

$$\frac{\partial P}{\partial N} = F_2(a_2 N^{a_2-1} - a_1 N^{a_1-1}) + \Delta F a_1 N^{a_1-1}. \tag{3.14}$$

From (3.13) and (3.14) we can get

$$N^* = [\frac{\Phi(d_+^2)}{\Phi(d_+^1)}]^{\frac{1}{a_2 - a_1}} > 1. \tag{3.15}$$

And then we compute

$$P^* = [\frac{\Phi(d_+^2)}{\Phi(d_+^1)}]^{\frac{a_2}{a_1 - a_2}} \left[F_2 - F_1 \frac{\Phi(d_+^2)}{\Phi(d_+^1)}\right] = Ee^{-r\tau}[\frac{\Phi(d_+^2)}{\Phi(d_+^1)}]^{\frac{a_2}{a_1 - a_2}} \frac{\Phi(d_-^1)\Phi(d_+^2) - \Phi(d_+^1)\Phi(d_-^2)}{\Phi(d_+^1)} > 0.$$
(3.16)

This ends the proof of Theorem 3.1.

Remark 3.2: Although the paper is built on the dependence between the options price and the number of shares, options being independent trading instruments are detached from their hedging role. That is one can trade only on options F'_1 and F'_2 to realize an arbitrage which is more than P^*.

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