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A New Version of the Exact Individual Trajectories Method for the Valuation of Long-Term Care Insurance

Massimo Angrisani[1] and Emanuela Ostili[2]

[1]*University of Rome La Sapienza, 00161 Roma, Via del Castro laurenziano 9, E-mail: massimo.angrisani@uniroma1.it*
[2]*IVASS, Italian Prudential Supervisor, 00184 Roma, Via del Quirinale 21, E-mail: emanuela.ostili@ivass.it*

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Abstract: *In the article a new version of the method called Exact Individual Trajectories Method (EIT),used for the management of pension funds, is introduced and tested for quantification of the annual premium need to finance a long-term care insurance system. This method developed on an axiomatic basis and is an alternative to the already known multiple state valuation models based on Markov and semi-Markov stochastic processes and methods of actuarial present value. The EIT is formulated on an individual basis which takes into account, for a hypothetical insured person, the set of all possible future life events, called feasible trajectories. In the articlesome numerical results are also presented.*

Keywords: *long-term care system; ageing; insurance protection products.*

1. INTRODUCTION

The main factors characterising Italian (and European) population trends from the socio-demographic point of view, including in the first place demographic ageing, but also the decline of the extended family with a rise in the rate of working women, are prompting growing interest in LTC insurance policies, and greater use of them can be expected in the future.

Inthe first part of this paper we introduce a new version of the *Exact Individual Trajectories Method (E.I.T.)*[1], a time-discrete actuarial model,on an axiomatic basis, introduced for the management of a pension fund. In the second part of the paper this new version is applied for the quantification of the cost of a long-term care insurance policy (standalone annuity) in terms of the individual equilibrium premium. The method takes into account thesetof all possible future life events of the insured person's position, i.e. the feasible trajectories. These trajectories are represented by vectors formally expressed in term of states assumed by the insured person and of the permanence time in each state, i.e. the duration. To each trajectory corresponds a single probability in Markovian hypothesis.

The components of each trajectory are transformed interms of financial cash-flow: paid premiums by the insured person and care treatment by insurance concern.

With aggregation of discounted cash flow vectors with relative probabilities it is possible to assess the individual equilibrium premium.

With the E.I.T. actuarial model, and also assuming that the condition of non-self-sufficiency cannot reverse but only, if anything, get worse, we can describe, and thus consider for the purposes of economic assessment, all the possible life stories of the individual, weighted according to the relative probability of their coming about. We can also determine the cash flow for each trajectory. The models for assessment of long-term care insurance products and, in general, disability and related insurance (which includes for example, permanent health insurance and critical illness insurance) to be found in the literature are essentially of the multiple-state type in which the trend of the insured person's position is governed by a Markovian stochastic process. In these models we can, with Monte Carlo simulation, determine a series of possible outcomes; the robustness of the results obtained depending on the number of simulations performed. With an increasing number of simulations the stability and mean value of the outcomes converge towards the "true" value. In the case of E.I.T., given the technical assumptions, the actuarial values take into account all the life stories of individuals weighted according to the relative probability of occurrence; thus, there is no uncertainty deriving from use of a simulation process. It is also is an alternative to methods of actuarial present value in which the results are provided in an aggregated manner.

Below a modified version of the E.I.T.is proposed, and in section 3 an application for assessment of long-term-care insurance coverage.

2. NEW VERSION OF E.I.T.

2.1. Notation and corporative S_i states

In the exact individual trajectory model the primary need is to define the life-cycle of the insured person whose position is, in each year, specified by the states it can take on (e.g. active/contributor, beneficiary of insurance/ social security supplied on the occurrence of certain events which depend on the type of risk taken on by the insurer or social security system concerned). One and only one state is to be assumed in each year.

We indicate with:

– X: age of the insured person at the start of assessment;

- $(\omega - 1)$: maximum age insured person can reach;
- $S \equiv \{S_1, S_2, S_3 \dots S_L\}$: the set of states that can be entered upon by the insured person;
- S_1: the state at the start of assessment;
- S_L: the state corresponding to elimination of the position.

Given the set of natural numbers $M_1 \equiv \{1,2,\dots\dots, L\}$ and the set of states $S \equiv \{S_1, S_2, S_3 \dots, S_L\}$, we hypothesise an order states induced by the natural numbers defined thus:

$$\forall h, k \in M_1 : S_h \leq S_k \Leftrightarrow h \leq k \tag{1}$$

The above order shows the reflexive, symmetric and transitive properties and is, moreover, complete, insofar as the above relation is defined for all the pairs considered, or in other words there are no pairs of non-comparable states $\in S$ for which the above relation is not defined.

2.2. Feasible trajectory identification: definition and axiomatic basis

The trajectory is considered as the vector for $\omega - x + 1$ components in which the i^{th} component is the state taken on by the position of the insured person after i years and i_t is the index corresponding to the terminal component of the trajectory, or in other words the component in which S_L first appears with $i_t \leq \omega - x$.

A feasible trajectory is an application $\Pi(\cdot)$ of the set of natural numbers $M \equiv \{0,1,2,\dots\dots, \omega\text{-}x\}$ to the ordered set of states (S)that satisfies the following two axioms A_1 e A_2.

Based on axiom $\mathbf{A_1}$:

$$\forall i, j \in M \ se \ i < j, \quad \Pi(i) = S_h e \Pi(j) = S_k \Rightarrow h \leq k$$

Function $\Pi(\cdot)$ is, therefore, monotonous, non-decreasing from the set of natural numbers to the set of states ordered as defined in (1).

On the basis of axiom $\mathbf{A_2}$ function $\Pi(\cdot): M \rightarrow S$ is such that:

$\Pi(0) = S_1$ and $\Pi(\omega - x) = S_L$

in which S_L is the state of elimination of the position.

The above two axioms constitute the basis of the model as formulated in the present work.

It will be seen that the following proposition follows from axiom A_1.

if $i \in M$ e $\Pi(i) = S_L \Rightarrow \Pi(j) = S_L \quad \forall \ j: j \geq i$

2.3. Definition of the duration vector in the S states

In this section we introduce the notion of the vector of the permanence time in each state, i.e. the "duration in each state", there by modifying the original design of E.I.T.

Let us take the case of a feasible generic trajectory generated by function $\Pi(\cdot)$; we then go on to define the set of N_k indexes, for $k = 1,\dots$, L associated with each state taken on in this trajectory:

$$\begin{cases} N_1 = \{i \in M : \Pi(\cdot) = S_1\} \\ N_2 = \{i \in M : \Pi(\cdot) = S_2\} \\ N_3 = \{i \in M : \Pi(\cdot) = S_3\} \\ \dots\dots\dots\dots\dots\dots \\ N_L = \{i \in M : \Pi(\cdot) = S_L\} \end{cases} \tag{2}$$

It can readily be verified that the N_k, set for $k = 1,\dots$L as defined by (2) show the following properties.

Properties of the N_k sets

a) The first and last set are nonempty.
This property follows from axiom A_2.

b) The generic N_k set, if nonempty, is an interval of natural numbers that can degenerate in one point alone.
In fact, for axiom A_1 it turns out that:
if $i, j \in M$ with $i \le j$ and $\Pi(i) = \Pi(j)$ then for $\forall\ h \in M$ and $i \le h \le j$ it is $\Pi(i) = \Pi(h) = \Pi(j)$;
Then for $\forall\ N_k$(nonempty), for $k = 1,\dots L$, $\forall\, i \in N_K$ we thus have that:
$\min_{i \in N_K} i \le i \le \max_{i \in N_K} i$

c) Given the indexes i and j, if $i \in N_h$ and $j \in N_k$ and $h < k \Rightarrow i<j$ This property follows from axiom A_1 and from (2)

d) $\bigcup_{h=1}^{L} N_h = M$

It is demonstrated verifying that the first set is contained within the second and vice versa, or in other terms:

$$\forall\, h, \text{ if } i \in N_h \text{ then } i \in M \text{ for} \tag{2}$$

$\forall$ i, if $i \in M$ then a state of axiom S is associated with it and, thus, through the association given by [2] there exists one and only one N_h set to which the above index i belongs.

e) $\forall\, i, j \in M\; Ni \cap N_j$ =empty set.

This property follows from property b). In fact, if one of the two were empty, the intersection would certainly be an empty set; if both were non empty, they could have no points in common because it would mean that they contemporaneously take on two different states, contrary to the design of the model. We now go on to definition of vector t of duration in the states associated with the same trajectory in the following recursive way:

$$\begin{cases} t_1 = \max\limits_{i \in N_1} i \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ t_2 = \max\limits_{i \in N_1 \cup N_2} i - t_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ t_h = \max\limits_{i \in N_1 \cup N_2 \ldots \cup N_h} i - t_1 - \ldots - t_{h-1}\; h = 1, \ldots\ldots L-1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ t_{L-1} = i_t - t_1 - t_2 \ldots - t_{L-2} - 1 \end{cases} \tag{3}$$

- t_1 = duration in the first state;
- t_2= duration in the second state;
- t_h= duration in the h^{th} state;
- t_{L-1}= duration in state (L–1);
- i_t = index corresponding to the terminal component of the trajectory in which state S_L first appears, then $i_t \in N_L$.

We now go on to demonstrate that $t_h \geq 0$ per $\forall\, h = 1, \ldots, L-1$

From the last equality relation (see above) we derive: $\sum\limits_{i=1}^{L-1} t_i = i_t - 1$

The generic trajectory being considered a feasible trajectory, set N_1 is certainly nonempty and $\min\limits_{i \in N_1} \mathrm{i} = 0$ is thus, given (3) $t_1 \geq 0$.

As for t_2 is set N_2 is empty, it comes to 0; in fact:

$\max\limits_{i \in N_1 \cup N_2} \mathrm{i} = \max\limits_{i \in N_1} \mathrm{i}$ and thus for (3):

$$t_2 = \max_{i \in N_1 \cup N_2} \mathrm{i} - t_1 = \max_{i \in N_1} \mathrm{i} - \max_{i \in N_1} \mathrm{i} = 0$$

otherwise, if N_2 is nonempty, it turns out that

$$t_2 = \max_{i \in N_1 \cup N_2} i - t_1 = \max_{i \in N_1 \cup N_2} i - \max_{i \in N_1} i$$

and, for the property c) of the N_k: sets $t_2>0$. Thus, in general, $t_2 \geq 0$

$$t_3 = \max_{i \in N_1 \cup N_2 \cup N_3} i - t_1 - t_2 = \max_{i \in N_1 \cup N_2 \cup N_3} i - \max_{i \in N_1 \cup N_2} i + t_1 - t_1$$

If set N_3 is empty t_3 comes to 0; in fact:

$$\max_{i \in N_1 \cup N_2 \cup N_3} i = \max_{i \in N_1 \cup N_2} i$$

Otherwise, if N_3 is nonempty, the result is, for property c) of sets N_k: $t_3>0$. Thus $t_3 \geq 0$.

And so forth; then $t_h \geq 0$ per $\forall\, h = 1,\ldots.., L–2$

From (3) we have:

$$t_{L-1} = i_t - t_1 - t_2 \ldots.. - t_{L-2} - 1 = i_t - t_1 - t_2 \ldots.. - t_{L-3} - \max_{i \in N_1 \cup N_2 \ldots \cup N_{L-2}} i + (t_1 + t_2 \ldots + t_{L-3}) - 1$$

Applying the appropriate simplifications, we obtain:

$$t_{L-1} = i_t - \max_{i \in N_1 \cup N_2 \ldots \cup N_{L-2}} i - 1$$

since $i_t \in N_L$ and the latter is certainly nonempty, and so, for property c) of sets N_k, we have:

$$i_t > \max_{i \in N_1 \cup N_2 \ldots \cup N_{L-2}} i$$

And thus $i_t - \max_{i \in N_1 \cup N_2 \ldots \cup N_{L-2}} i \geq 1$

And thus, $t_{L-1} \geq 0$

And so forth; thus, $t_h \geq 0$ for $\forall\, h = 1,\ldots.., L - 1$.

In the following section we demonstrate the correspondence by virtue of which a feasible trajectory can be transformed into a duration vector in the various states, as defined in (3).

2.4. Biunivocal correspondence between the feasible trajectories and the duration vectors in S states

Every trajectory is in biunivocal correspondence with the durations in each state. In fact, given a certain trajectory, the duration vector is uniquely determined by (2) and (3); thus for two different trajectories, differing even in one single component, they correspond to two distinct duration vectors. Given the duration vector, applying (3) the lower and upper indexes of each N_K set for $k = 1,\ldots L$ are determined univocally by virtue of the properties of the N_K sets and axiom A_1; it is then possible, by means of (2), to identify the corresponding feasible trajectory univocally.

We now identify set $Q \subseteq \mathbf{R}^{L-1}$ consisting of points $P(Z_1, Z_2,.. Z_{i\ldots}\ Z_{L-1})$ - with integer coordinates - which verify the following conditions:

$$\begin{cases} 1)\quad Z_i \text{ interi} \geq 0 \text{ per } i = 1,2\ldots \text{(L-2)} \\ 2)\quad Z_1 + Z_2 + + Z_{L-1} \leq \omega - x \\ 3)\quad Z_{L-1} \geq 1 \end{cases} \tag{4}$$

The relation that associates points Q with the duration vector in S states is defined by (5), thus:

$$\begin{cases} Z_1 = t_1 \\ Z_2 = t_2 \\ \ldots. \\ Z_h = t_h h = 1,\ldots L-1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ Z_{L-1} = t_{L-1} + 1 \end{cases} \tag{5}$$

The **representation theorem** states that each feasible trajectory is in biunivocal correspondence to a point in set Q identified by (4).

The association of trajectory with each point $P(Z_1, Z_2,.. Z_{i\ldots}\ Z_{L-1}) \in Q$ corresponds to an application $\Pi[P(Z_1, Z_2,.. Z_{i\ldots}\ Z_{L-1})](\cdot)$: M S;

In the first place, it is readily demonstrated that point $P(Z_1, Z_2,.. Z_{i\ldots} Z_{L-1})$ whose coordinates are defined by (5) belongs to Q - or in other terms (4) applies - as will be argued below.

Since $t_h \geq 0$ applies for $\forall\ h = 1,\ldots.., L-1$:

$z_h \geq 0$ for $\forall\ h$=1,.....,L-2 $e Z_{L-1} \geq 1$.

Moreover, by virtue of (5) we have: $Z_1 + Z_2 + \ldots\ldots. Z_{L-1} = \sum_{h=1}^{L\ 1} t_h + 1$ which, given the last relation of (3) is equal to i_t.

Since $i_t \leq \omega - x$ we have $Z_1 + Z_2 + \ldots\ldots. Z_{L-1} = i_t \leq \omega - x$ **Q.E.D.**

The coordinates of each point coincide with the components of the duration vector with the exception of the last component, which differs by one unit, and so each point is in biunivocal correspondence to the duration vector. Thus, given their transitive property points $\in Q$ are also in biunivocal correspondence to the feasible trajectories.

In operational terms, association of the trajectory with each point is possible, identifying first the corresponding duration vector and then, as explained above, going on to the corresponding feasible trajectory. Or in reverse, given the generic feasible trajectory it is possible through (2) and (3) to identify the duration vector in the states and with (5) the corresponding point $\in Q$.

The set of points $P(Z_1, Z_2, .. Z_{i...} Z_{L-1})$ Q thus indicates the range of possible trajectories. **QED**

2.5. Probabilities attributed to the feasible trajectories

Once all the possible "life stories" (feasible trajectories) of the insured person have been defined, it is then necessary to go on to determine the relative probabilities of occurrence in Markovian hypotheses. To this end, the matrixes $B(i)$ are defined that indicate for the feasible trajectories the probabilities of remaining/transition between states – in the i^{th} year – meaning by *probability of transition* the probability of the insured person moving on, at a certain age, from one generic state S_K to another, and by *probability of remaining* the probability of the person remaining in the same state S_k.The above-mentioned matrixes (see the table n. 1) indicate, both in the headings of the columns and in those of the rows, all the possible states belonging to the set of S states. At the intersection between each row and each column can be seen the probability of remaining/transition between the state the insured person took on in the i^{th} year and the state the same person may take on in the year immediately following.

Table 1
Matrix of probabilities of remaining/transition.

		state in the year i+1							
		S_1	S_2	...	S_h	...	S_k	...	S_L
state in the year i	S_1	${}^1p_{2,3...L}(x+i,x+i+1)$	${}^1q_{2,}^{3...L}(x+i,x+i+1)$	...	...	...	${}^1q_k^{2,3...L}(x+i,x+i+1)$	...	${}^1q_L^{2,3...L}(x+i,x+i+1)$
	S_2	0	${}^2p_{3...L}(x+i,x+i+1)$	...	${}^2q_h^{3...L}(x+i,x+i+1)$	...	${}^2q_k^{3...L}(x+i,x+i+1)$	...	${}^2q_L^{3...L}(x+i,x+i+1)$
	...	0	0	...	...	...	...	...	...
	S_h	0	0	0	${}^hp_{h+1...L}(x+i,x+i+1)$	...	${}^hq_k^{h+1...L}(x+i,x+i+1)$	...	${}^hq_L^{h+1...L}(x+i,x+i+1)$
	S_k	0	0	0	0	0	${}^kp_{k+1...L}(x+i,x+i+1)$	...	${}^kq_L^{k+1...L}(x+i,x+i+1)$
	...	0	0	0	0	0	0	...	...
	S_L	0	0	0	0	0	0	0	1

Indicated in each cell of the matrix are the probabilities of remaining/

transition between the state taken on in year i (indicated for each row) and the state taken on in year i+1 (indicated for each column).

These matrixes have the following properties:

- the elements of the matrixes are non negative and less than or equal to 1;
- the sum of elements in each row of the matrixes amounts to 1;
- the matrixes are upper triangular: this is due to the order of rows and columns determined by the order of states;
- given the initial condition defined by axiom A_2, matrix $B(0)$ consists solely of the first row of generic matrix B(i);
- given the final condition defined by axiom A_2 matrix B(ω-x-1) shows all the zero elements except for the elements in the last column (column S_L), which amount to 1;
- given the order of states, the probability of remaining in state S_L $\forall i = 1,...$ (ω–x–1) is 1.

The probability of remaining/transition for each feasible trajectory, represented by the vector of states Π(0), Π(1),…Π(ω-x) obtained as function Π (·): $M \rightarrow S$ which verifies the axiomatic basis of model comes (in Markovian hypothesis) to:

$$\prod_{i=0}^{w\ x\ 1} \text{prob}\{\Pi(i), \Pi(i+1)\}$$

in which the probability of remaining/transition $prob\{\Pi(i), \Pi(i+1)\}$ is defined by matrix $B(i)$.

The sum of the probabilities for all the feasible trajectories relative to the position of the insured person amounts to 1, i.e. $\sum_i \text{prob}_{\text{tam}_i} = 1$

2.6. Transformation of feasible vectors' trajectories from state vectors to (contributions/benefits) flow vectors

Transformation of the feasible trajectories from state vectors to contributions/benefits flow vectors is accomplished taking into account the regulations (hypothesised in the present paper) applied to insurance coverage by virtue of which each state can be attributed with the financial flow. For example, in application of an LTC coverage model, presented in section 3, in the case of self-sufficiency a premium will be paid by the insured person to the insurance company, while on the other hand, in the case of non-self-sufficiency, the insurance company will supply a certain benefit to the insured person.

3. THEORETICAL APPLICATIONOF THE NEW VERSION OF E.I.T.FOR LTC INSURANCE ASSESSMENT

3.1. Description of the typical coverage

The LTC insurance coverage hypothesised for the purposes of the present paper guarantees three forms of non-self-sufficiency in the performance of normal everyday activities (such as activities regarding personal hygiene, nutrition, mobility and household chores) increasing gravity- respectively "*level-1 non-self-sufficiency*", "*level-2 non-self-sufficiency*" and "*level-3 non-self-sufficiency*". Should the loss of autonomy be ascertained, the insurance coverage provides for three annuities of increasing proportions in relation to the degree of non-self-sufficiency (subject to yearly reassessment on the basis of a predetermined rate). The financing of the insurance coverage consists of a premium the insured person pays up to a preestablished age. However, the occurrence of some form of non-self-sufficiency entailing the right to welfare benefit also brings an end to the obligation to pay the premium, even if the insured person has not reached the maximum to qualify according to the policy. No other case apart from the occurrence of non-self-sufficiency is taken into consideration for interruption of payment of the premium.

We now come to application of the model presented in section 2 to LTC insurance coverage.

The range of states is given by the set $S = \{A, P_{I°}, P_{II°}, P_{III°}, E\}$ in which $S_1 = A$ = "*Self-sufficient*"; $S_2=P_{I°}$="*Level-I Non-Self-Sufficient*" which qualifies for a "Level-1 LTC benefit"; $S_3 = P_{II°}$= *Level-2 Non-Self-Sufficiency* which qualifies for "Level-2 LTC Benefit", $S_4 = P_{III°}$= *Level-3 Non-Self-Sufficiency* which qualifies for "Level-3 LTC Benefit" and S_5=E = *Elimination of the insured person's position on death*.

With the above association the order of (1) corresponds to the following relation:

$$A \le P_{I°} \le P_{II°} \le P_{III°} \le E \quad (1)$$

The order of states given in (1) implies that regression from the state of non-self-sufficiency is not feasible: thus, once a certain level of gravity of non-self-sufficiency is reached it is only possible to remain in the same state of non-self-sufficiency until death or succumb to a more serious degree of non-self-sufficiency. The hypothesis of recovery is therefore ruled out; in other words, the probability of recovery, i.e. the possibility that having once reached a form of non-self-sufficiency – level I, level II or level III – the insured person may be cured and reacquire autonomy in the performance of normal everyday activities (i.e. the probability of moving from $P_{I°}$ to A, from $P_{II°}$ to $P_{I°}$ or A and from $P_{III°}$ to $P_{II°}$, $P_{I°}$ or A) is zero.This assumption is dictated by the need to simplify the design of the model – both from the formal and the operational point of view – but above all by the lack of statistical evidence to determine such probabilities to a reasonably reliable degree.

Below we present a scheme of the insured person's life cycle.

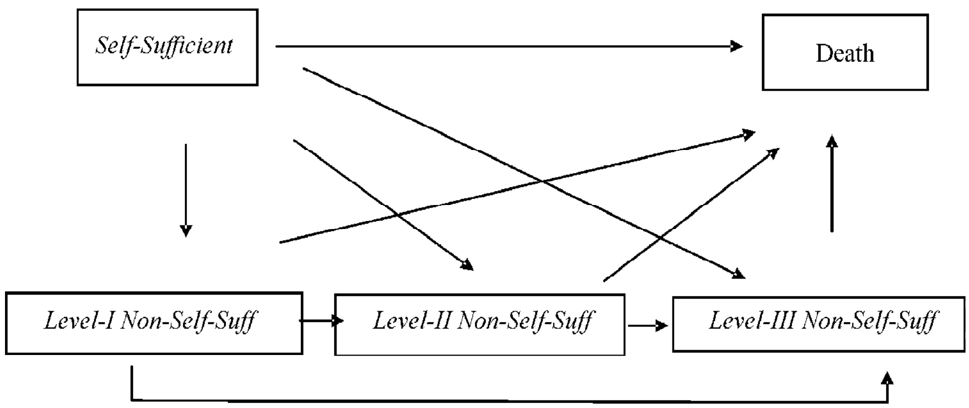


Figure 1: Scheme of the insured person's life cycle

We state two axioms:

(A_1) function $\Pi(\cdot) : M \rightarrow S$ is not decreasing, i.e. if $i, j \in M$ and $i \leq j \Rightarrow \Pi(i) \leq \Pi(j)$;

(A_2) $\Pi(0) = A$ and $\Pi(\omega\text{-}x) = E$.

We now come to the proposition deriving from axiom $A_{1.}$

$$\text{If } i \leq M \text{ and } \Pi(i) = E \Rightarrow \Pi(j) = E \quad \forall j: i \leq j$$

We now define the sets of indexes N_k, for $k = 1,\ldots, 5$ associated with each state taken on in the feasible generic trajectory generated by application $\Pi(\cdot)$:

$$\begin{cases} N_1 = \{i \in M : \Pi(\cdot) = \mathrm{A}\} \\ N_2 = \{i \in M : \Pi(\cdot) = \mathrm{P}_{I°}\} \\ N_3 = \{i \in M : \Pi(\cdot) = \mathrm{P}_{II°}\} \\ N_4 = \{i \in M : \Pi(\cdot) = \mathrm{P}_{III°}\} \\ N_5 = \{i \in M : \Pi(\cdot) = \mathrm{E}\} \end{cases} \tag{2}$$

We now go on to identify the durations in the states associated with the trajectory under consideration thus:

$$t_2 = \begin{cases} t_1 = \max_{i \in N_1} i \quad t_2 = \max_{i \in N_1 \cup N_2} i - t_1 \\ \max_{i \in N_1 \cup N_2} i - t_1 \\ t_3 = \max_{i \in N_1 \cup N_2 \cup N_3} i - t_1 - t_2 \\ t_4 = i_5 - t_1 - t_2 - t_3 - 1 \end{cases} \tag{3}$$

i_5 is the index corresponding to the terminal component of the trajectory in which state E first appears.

We then identify the set $Q \subseteq \mathbf{R}^3$ consisting of the points $P(Z_1, Z_2, Z_{3,} Z_4)$ - with integer coordinates - that verify the following conditions:

$$\begin{cases} 1) z_i \text{ interi } \geq 0 \text{ per } i = 1,2,3 \\ 2) z_1 + z_2 + z_3 + z_4 \leq \omega - x \\ 3) z_4 \geq 1 \end{cases} \tag{4}$$

Let:

$$\begin{cases} Z_1 = t_1 \\ Z_2 = t_2 \\ Z_3 = t_3 \\ Z_4 = t_4 + 1 \end{cases} \tag{5}$$

Point $P(Z_1, Z_2, Z_3 Z_4)$, whose coordinates are defined by (4), belongs to set Q which represents the set of points on the plane identifying all the feasible trajectories.

The figure 2 shows the set of points $P(Z_1, Z_2, Z_3) \in$ which identify the feasible trajectories under the hypothesis that there are only two levels of non-self-sufficiency rather than three in order to represent them in three-dimensional space.

The figure shows on axis Z_1 the first coordinates of the point corresponding to the duration in the state of self-sufficiency, on axis Z_2 the second coordinate of the point corresponding to the duration in the state of level-1 non-self-sufficiency, and on axis Z_3 the third coordinate of the point corresponding to duration in the state of level-2 non-self-sufficiency.

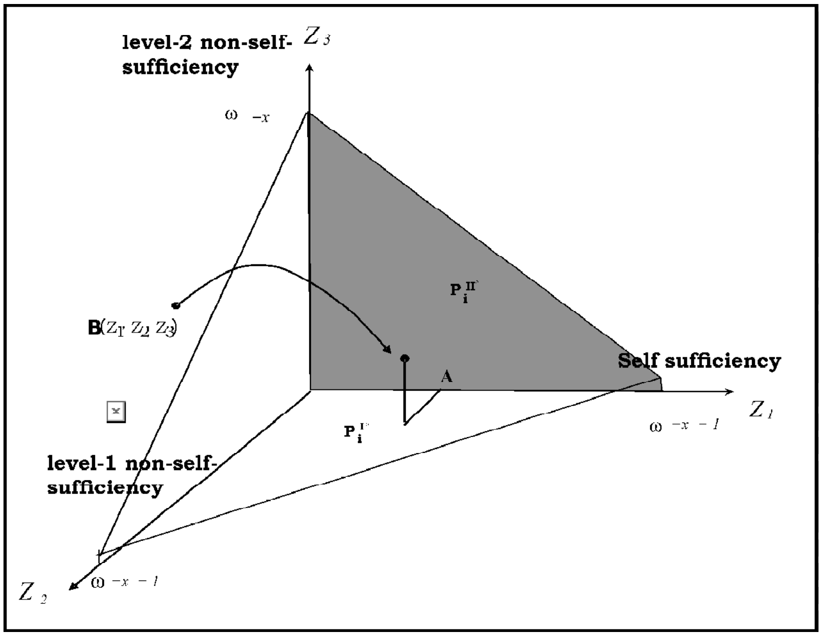


Figure 2: the set of points $P(Z_1, Z_2, Z_3)$ Q that identify the feasible trajectories

3.2. Probabilities of feasible trajectories

The table n. 2 represents the matrix of remaining/transition probabilities of the feasible trajectories B(i) with the same significance as indicated in table 1 of section 2.5.

Verification that the sum of the probabilities of all the feasible trajectories amounts to 1 was performed with the program with which simulations of the model were carried out as illustrated in the section 4.

Table 2
Matrix of the remaining/transition probabilities of the feasible trajectories B(i) for LTC insurance assessment

		STATE IN THE YEAR i+1				
		A	P_{I°	P_{II°	P_{III°	E
STATE IN THE YEAR i	A	${}^{A}p_{I^\circ,II^\circ,III^\circ,D}(x+i,x+i+1)$	${}^{A}q_{I^\circ}{}^{II^\circ,III^\circ,D}(x+i,x+i+1)$	${}^{A}q_{II^\circ}{}^{I^\circ,III^\circ,D}(x+i,x+i+1)$	${}^{A}q_{III^\circ}{}^{I^\circ,II^\circ,D}(x+i,x+i+1)$	${}^{A}q_{D}{}^{I^\circ,II^\circ,III^\circ}(x+i,x+i+1)$
	P_{I°	0	${}^{I^\circ}p_{II^\circ,III^\circ,D}(x+i,x+i+1)$	${}^{I^\circ}q_{II^\circ}{}^{III^\circ,D}(x+i,x+i+1)$	${}^{I^\circ}q_{III^\circ}{}^{II^\circ,D}(x+i,x+i+1)$	${}^{I^\circ}q_{D}{}^{II^\circ,III^\circ}(x+i,x+i+1)$
	P_{II°	0	0	${}^{II^\circ}p_{III^\circ,D}(x+i,x+i+1)$	${}^{II^\circ}q_{III^\circ}{}^{D}(x+i,x+i+1)$	${}^{II^\circ}q_{D}{}^{III^\circ}(x+i,x+i+1)$
	P_{III°	0	0	0	${}^{III^\circ}p_{D}(x+i,x+i+1)$	${}^{III^\circ}q_{D}(x+i,x+i+1)$
	E	0	0	0	0	1

Detailed description of the elements indicated in the above table is provided in attachment 1.

3.3. Transformationof the feasible trajectories from state vectors to (contributions/benefits) flow vectors

The first point to make here is that *self-sufficient* persons are under the obligation to pay the premium until they reach a certain age, which is predetermined. In the E.I.T. model, as designed for the present purposes, the condition of self-sufficiency is not distinguished between contributors and non-contributors who have reached the maximum age for payment of the premium. Thus age control has been included in the programming, at a stage of transformation of feasible trajectories from state vectors to premiums/benefits flow vectors for this purpose. This control entails the condition that if, in generic year i, the *self-sufficient* insured person is of an age of (x+i), above the maximum age predetermined for payment of the premium (set at 65 years in the case of the simulation) the contribution due is zero. Including an age limit does not change the outcomes of the theoretical basis of the model demonstrated in the present paper.

4. ASSESSMENTS

In this section we illustrate application of the E.I.T.model to a hypothetical case of LTC insurance coverage against the risk of non-self-sufficiency. Given the paucity of the data that might be utilised for reliable assessment of the probabilities of becoming non-self-sufficient at level I, level II and level III, and of the probabilities of transition from one state of non-self-sufficiency to another, as pointed out below, this simulation has the sole purpose of providing numerical exemplification of application of the model, as formalised in the previous chapter.

Also to be borne in mind in evaluating the results is the hypothesis that the LTC insurance is stipulated in a certain year that coincides with the starting year of the simulation. Obviously, the simulation extends over a span of time that covers the entire insurance life cycle.

The aim of the simulation, then, is to determine the contribution financing the guaranteed insurance coverage, or in other words the individual equilibrium premium – the sum that, on the basis of the hypotheses employed, makes the present expected value of the insurance benefits equal to the present expected value of the premiums received by the insurance concern guaranteeing the policy.

4.1. Contract conditions

The LTC insurance coverage hypothesised covers three forms of non-self-sufficiency in the performance of normal everyday activities with increasing levels of gravity: "level-1 non-self-sufficiency", "level-2 non-self-sufficiency" and "level-3non-self-sufficiency". Should the loss of autonomy be recognised, three levels of LTC benefits are attributed in terms of annuities to be reassessed annually on the basis of inflation.

Payment of the premium is to be made up to a predetermined age.

The LTC benefits (1st, 2nd and 3rd level) are in any case supplied, even if the event invalidating the insured person occurs before reaching the maximum age set for payment of the premium. In this case, the occurrence of (1st, 2nd and 3rd level) non-self-sufficiency puts an end to the obligation to pay the premium.

Guaranteed benefits

The guaranteed LTC benefits in the first year of simulation (coinciding with the first year of insurance coverage) amount to the following sums:

- 10,200 Euro per year (corresponding to 850 Euro per month) in the case of level-1 LTC benefits;
- 16,800 Euro per year (corresponding to 1,400 Euro per month) in the case of level-2 LTC benefits;
- 18,000 Euro per year (corresponding to 1,500 Euro per month) in the case of level-3 LTC benefits.

As previously pointed out, the above sums are reassessed annually on the basis of the rate of inflation hypothesised in the projections.

Contribution

The financing of insurance coverage derives from an annual premium paid by the insured person (at the beginning of each year), which can be reassessed annually on the basis of the reassessment rate hypothesised in the projections, which constitutes, as equilibrium premium, the result of the simulations described above.

The maximum age at which the insured person has the obligation to pay the premium stands at 65 years.

4.2. The technical hypotheses applied

a) Rate of reassessment of the premiums and guaranteed benefits: the rate of reassessment hypothesised stands at 1% per year;

b) demographic hypotheses: Self-Sufficient Survival Table: ISTAT 2014/M;

c) probabilities of death of level-1 non-self-sufficient: equal to the probability of death of self-sufficient person increased by 10%;

d) probabilities of death of level-2 non-self-sufficient person: equal to the probability of death of self-sufficient person increased by 20%;

e) probabilities of death of level-3 non-self-sufficient person: equal to the probability of death of self-sufficient person increased by 27%;

f) probabilities of self-sufficient becoming level-1 non-self-sufficient: probabilities set in such a way they are similar to the trend of those derived from those used by S.Haberman and E.Pitacco for some numerical exemplifications of a policy of the stand-alone annuity type, drawn from data of the UK Office of population Censuses and Surveys (OPCS) (Haberman S. and Pitacco E., 1999);

g) probabilities of the self-sufficient becoming level-2 non-self-sufficient: probabilities set in such a way they are similar to the trend of those derived from those used by S.Haberman and E.Pitacco for some numerical exemplifications of a policy of the stand-alone annuity type, drawn from data of the UK Office of population Censuses and Surveys (OPCS)(Haberman S. and Pitacco E., 1999). There follows graphic representation of the said probabilities (graph n.1);

h) probabilities of the self-sufficient becoming level-3 non-self-sufficient: they are equal to the probabilities of the self-sufficient becoming level-2 non-self-sufficient.

i) probabilities of transition from level-1 state of non-self-sufficiency to level 2: the equal to the probabilities of a self-sufficient becoming level-2 non-self-sufficient;

j) probabilities of transition from level-1 state of non-self-sufficiency to level 3: the equal to the probabilities of a self-sufficient becoming level-3 non-self-sufficient;

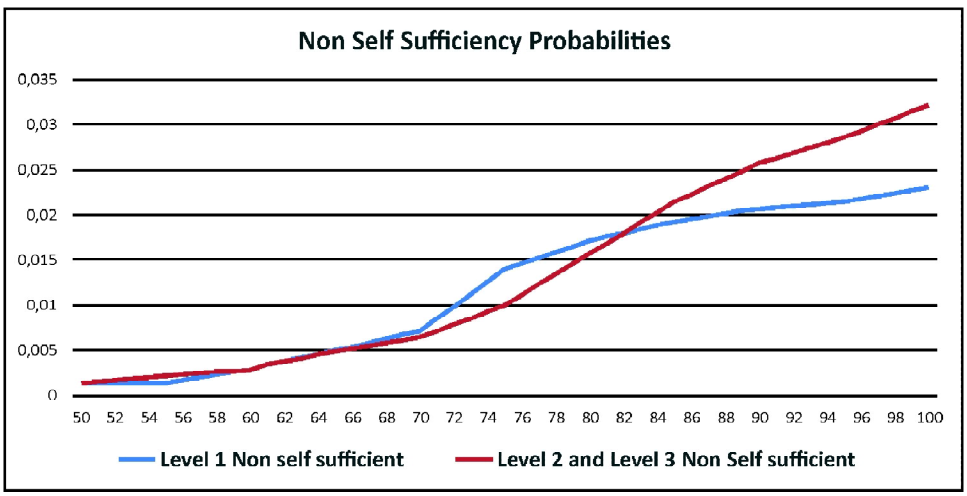
k) probabilities of transition from level-2 state of non-self-sufficiency to level-3 state of non-self-sufficiency: the equal to the probabilities of the self-sufficient becoming level-3 non-self-sufficient;

l) technical actualisation rate. In the simulations, to actualise the financial flows we applied atechnical rate of 2%.

m) Insurance management costs: for the sake of simplicity, no charge on the premium to cover the insurance management costs is considered.

For the cases under study here, the probability of recovery (i.e. the possibility that, having once succumbed to a form of non-self-sufficiency, whether level 1 or level 2, 3 the insured person can be cured to the extent of re-acquiring autonomy in the performance of normal everyday activities) is set at zero. This assumption, apart from simplifying the design of the model – from both the formal and operational point of view, through the creation of the simulation program – is due above all to the lack of an effective relevant database upon which reliable hypotheses could be formulated.

Also assumption of the above hypotheses regarding the probabilities of becoming non-self-sufficient is due mainly to the lack, so far, of reliable data to measure the risk of losing self-sufficiency.



Graph 1

4.3. Results

Indicated in table 3 is the equilibrium premium for an individual stipulating LTC insurance at the age of 50 determined on the basis of the technical hypotheses illustrated above.

It is to be borne in mind that payment of the premium ends if the insured person becomes non-self-sufficient at level 1, 2 or 3. Ceteris paribus, obviously, given the distribution of inflows and outflows over time, the sum of the equilibrium premium is decreasing with respect to the technical actualisation rate and increasing with respect to the age at stipulation of the policy and the sum of benefits acquired.

Table 3
Individual equilibrium premium

Registered Age	*Age Limit for Payment of Premium*	*Technical Rate of Actualisation*	*Individual Premium*
50 Years	65 Years	2%	€ 5,570

5. CONCLUSIONS

With the E.I.T.actuarial model it is, then, possible to describe, and so consider for the purposes of economic assessment, all the possible life stories of the individual, weighted for the relative probabilities of occurrence. Given the technical hypotheses assumed, the result is not affected by the uncertainty which, by contrast, characterises processes based on simulation methods.

Moreover, a great deal of detailed data are available; for example, it is possible to access the cash-flows associated with each trajectory.

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Attachment 1: Elements of the matrix ofremaining/transition between states probabilities

${}^{A}p_{M,I°,II°,III°}(x+i, x+i+1)$ = probability of the insured person remaining self-sufficient between age (x+i) and (x+i+1);

${}^{I°}p_{M,II°,III°}(x+i, x+i+1)$ = probability of level-1 non-self-sufficient remaining in the same state between age (x+i) and (x+i+1);

${}^{II°}p_{M,III°}(x+i, x+i+1)$ = probability of level-2 non-self-sufficient remaining in the same state between age (x+i) and (x+i+1);

${}^{III°}p_{M}(x+i, x+i+1)$ = probability of level-3 non-self-sufficient remaining in the same state between age (x+i) and (x+i+1);

${}^{A}q_{I°}^{M,II°III°}(x+i, x+i+1)$ = probability of the self-sufficient becoming level-1 non-self-sufficient between age (x+i) and (x+i+1);

${}^{A}q_{II°}^{M,I°,III°}(x+i, x+i+1)$ = probability of self-sufficient becoming level-2 non-self-sufficient between age (x+i) and (x+i+1);

${}^{A}q_{III°}^{M,I°,II°}(x+i, x+i+1)$ = probability of self-sufficient becoming level-3 non-self-sufficient between age (x+i) and (x+i+1);

${}^{A}q_{M}^{I°,II°,III°}(x+i, x+i+1)$ = probability of self-sufficient dying between age (x+i) and (x+i+1);

${}^{I°}q_{II°}^{M,III°}(x+i, x+i+1)$ = probability of level-1 non-self-sufficient becoming level-2 non-self-sufficient between age (x+i) and (x+i+1);

${}^{II°}q_{III°}^{M}(x+i, x+i+1)$ = probability of level-2 non-self-sufficient becoming level-3 non-self-sufficient between age (x+i) and (x+i+1);

${}^{I°}q_{M}^{II°,III°}(x+i, x+i+1)$ = probability of level-1 non-self-sufficient dying between age (x+i) and (x+i+1);

${}^{II°}q_{M}^{III°}(x+i, x+i+1)$ = probability of level-2 non-self-sufficient dying between age (x+i) and (x+i+1)

${}^{III°}q_{M}(x+i, x+i+1)$ = probability of level-3 non-self-sufficient dying between age (x+i) and (x+i+1).