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Claims Reserving in One Year Approach: An Integrated View of the Merz and Wüthrich Model with New Further Specifications*

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***Abstract**: The aim of this paper is to provide an integrate derivation of the fa-mous model of Merz and Wüthrich in [Merz and Wüthrich (2008b)]. The formula in classical statistics framework highlights new connections and in-termediate parameters that we call* small Greeks*, these parameters are defined by development year instead of accident year as in original work in [Merz and Wüthrich(2008b)]. Also step by step numerical investigations with R code will be presented.*

Keywords: *claims reserving, prediction error, claims development result, one year view.*

1. INTRODUCTION

Claims reserving is one of the main concerns for a non life insurance company. From the granularity that claim handle1rs face to the single item written in the balance sheet by the accountants, a lot of professionals are involved in this process (claim handlers, actuaries, internal and external auditors, risk managers, accountants, IT experts, directors). A good balance between theoretical assumptions and practical issues may be found in *Claims Reserving in General Insurance* data represents a hybrid, since composed of certain data (payments) and data estimated by the company (reserves) recorded in the various exercises. The use of incurred data can cause distortive effects on the estimate of the volatility of future payments, as the assumptions underlying the estimate of reserves could absorb or neutralize the increasing or decreasing trends characterizing the historical series of the payments.

Provided that the mentioned paper includes only the results without demonstrations as terminal point of the previous papers [Merz and Wüthrich (2007)] and [Wüthrich *et al.* (2008)], we present a complete and

* Remark that this article reflects the personal view of the authors and not necessarily that of their Institutions.

integrate derivation of the model in a classical statistical way (different from the bayesian approach. highlighting new connections with specific analysis of the formulas, we call small Greeks to differentiate them from Capital Greeks introduced in [Merz and Wüthrich (2008b)]. In [Bühlmann *et al.* (2009)] the model was analyzed and derived in a bayesian context, with a priori distribution hypothesis, through a recursive algorithm, highlighting how the model derived from the chain ladder, positions itself at a lower level in terms of results (due to linear approximation). This approach was applied to obtain successively other results [Merz and Wüthrich(2014), Merz and Wüthrich(2015)]. These small Greeks are dependent on development year and they are useful to evaluate the prediction error on size beyond the triangle, in case tail provision is still significant, through the extrapolation. Moreover we provide interpretation and graphical analysis.

We try to highlight the main contribution of this paper: we refer to several papers behind M&W model even if in these articles the theoretical framework was very fragmentary with different notations and with some error subsequently corrected. We therefore are showing also a very structured and model's derivation for students, researchers and practitioners. In the text the demonstrations, the results and the R-code are declined in a compound way in order to achieve a better theoretical understanding following the conceptual map in figure 1.

We see hence that thousands of people use M&W formula but few of them know its demonstration and still less may have studied the derivations. We want to overcome this situation.

The first stochastic chain ladder model namely the distribution free Mack model (in the time series version introducing conditional re-sampling technique as in [Merz and Wüthrich(2008a)]) was necessary as starting point.

The MW model has the objective to quantify the next year claims development result's volatility. At time I we evaluate the ultimate cost with the available information; at time $I + 1$ with more information this prediction will change. The difference between these predictions represents the claim development result (CDR) for the balance year $(I, I + 1]$. This value has a direct impact on profit and loss (P&L) and on solvency position of an insurance company.

Operationally in the model we analyze the prediction of CDR and the fluctuations (in terms of uncertainty) around that to answer to the following practical questions:

- in general we predict that CDR in the balance $(I, I + 1]$ in the budget plan written in I is 0 and we analyze uncertainty related to this prediction. So we define a prospective view about the potential deviations of the CDR from 0 in a specific volatility;

- In the P&L statement in $I + 1$ we observe the real CDR; we ask ourselves if this observation stays in a reasonable range around 0 or is an outlier. So we define a retrospective view and the volatility is decomposed in two components: process error and parameter error.

In claims reserving literature the run off uncertainty until the final extinction of the accident years has been of great interest. For the chain ladder method the theory below was formulated by [Mack(1993)]; this is the long term view that is important to investigate on financial strength. Indeed all the stochastic models for claims reserving proposed until now reflect this concern.

Nevertheless recently some works are concentrated on short term view as the one year Solvency II prospective because:

- the insolvency arrives before the claims closure;
- the short term view is relevant for management decisions taken on annual basis for the main balance sheet items;
- through the balance sheets the short term performance of the company is monitored by insurance supervisors, clients, investors, rating agencies, stock markets. This results impact on financial stability and reputation of the company in the insurance market.

In Section 2 we give a Chain Ladder's overview through the data organization, claims development results, basic concepts, times series approach and Merz, Wüthrich and Lysenko Lemma. In Section 3 we give a derivation of model [Mack(1993)]. In Section 4 we show the one-year view by means of [Merz and Wüthrich(2008b)] model. In Section 5 we provide the calculation of MSEP of CDR and ultimate view using the *small greeks*. In Section 6 the conclusion is given.

Table 1
Run-off triangle of cumulative payment

$i\,/\,j$	**0**	**1**	$\cdots$	j	$\cdots$	J
0	Y_{00}	Y_{11}	$\cdots$	Y_{1j}	$\cdots$	Y_{1J}
1	Y_{10}	Y_{11}	$\cdots$			
$\vdots$	$\vdots$					
i	Y_{i0}		Y_{ij}			
$\vdots$	$\vdots$					
I	Y_{I0}					

2. SECTION

2.1 Data Organization

Given Y_{ij} as the paid sum, with j as the delay in payment for accidents happened in the i-th year, usually called incremental payment. These payments are usually represented in the so-called run-off triangle (see Table 1).

Given instead $C_{ij} = \sum_{k=0}^{j} Y_{ik}$ as the cumulative payment, i.e., the sum paid-off for the i generation within the first J development years, the recursive relation $C_{ij} = C_{i,j-1} + Y_{ij}$ with $j > 0$ is effective. The ratio $F_{i,j-1} = C_{ij}/C_{i,j-1}$, named link ratio, is the factor connecting the cumulative payment between two close development years—the $j-1$ and the $j$for the same i generation. Assuming that the payment process of each generation will be surely over within J years, the overall cost of the i generation will be: $C_{i,J} = \sum_{k=0}^{J} Y_{ik}$ writing again the overall cost in the sum of the two addends will make things clearer:

$$C_{i,J}^{(t)} = \underbrace{\sum_{k=0}^{t-i} Y_{ik}}_{deterministic} + \underbrace{\sum_{k=t-i+1}^{J} Y_{ik}}_{stochastic}, \tag{1}$$

since in the t balance-sheet year the first addend is known for sure, while the second is subjected to randomness. For our purpose the balance-sheet year is $t = I$, so in the next we refer to I for meaning the time of evaluation. Therefore, the claims reserve estimate for the i generation for the t balance-sheet year, concerns the random component of $C_{i,J}^{(t)}$, that is to say, we have:

$$R_i^{(t)} = \sum_{k=t-i+1}^{J} Y_{ik} = C_{t-j,J}^{(t)} - C_{t-j,j}.$$

We will instead name $\hat{R}_i^{(t)} = \sum_{k=t-i+1}^{J} \hat{Y}_{ik}$ the claims reserve estimate, made at the t time, and $\hat{R}^{(t)} = \sum_{i+j>t} \hat{Y}_{ij}$ the overall claims reserve for all generations.

In the following passages, we will use t to make reference to the current date of estimate.

2.2 The Claims Development Result

The CDR is the technical result of the evolution of the claim settlement process. In other words, it calculates if the claims reserve R_i^I -set aside in the I balancesheet year, for the i generation—is enough to pay the claims $X_{i,t-i+1}$, between t and $t+1$ and to set aside the new claims reserve R_i^{I+1} in $I + 1$ formally:

$$CDR_{i,I+1} = R_i^I - (X_{i,I-i+1} + R_i^{I+1}) = C_{i,J}^I - C_{i,J}^{I+1}, \quad i = 1,\ldots,I \tag{2}$$

is a random variable if the observation moment is I, while it is a deterministic value if the observation moment is $I + 1$. In the risk estimate and solvency capital calculation framework, we are interested in I observation random variable, while, aspect observed in $I + 1$. Particularly, we have a loss if $CDR_{i,I+1} < 0$, while we have a gain with a positive result.

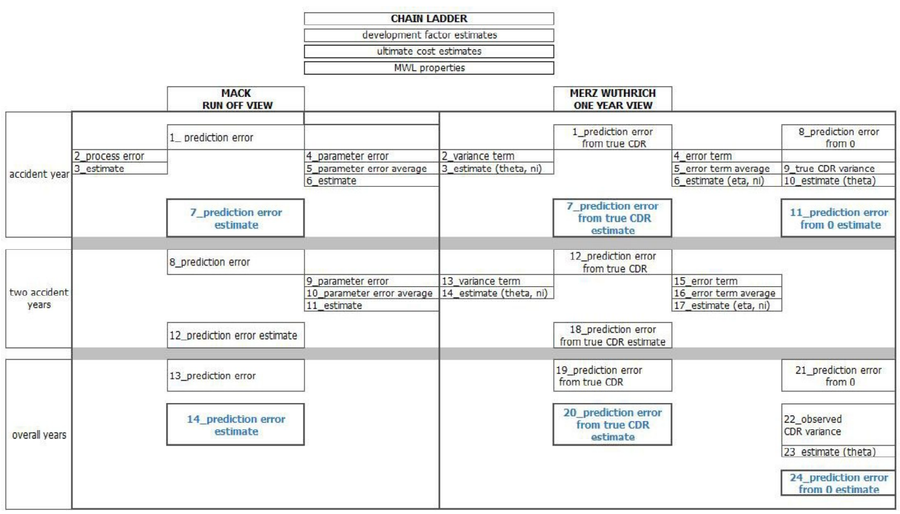

Figure 1: A map to drive in MSEP evaluation.

Provided that theoretical background of the model is complex enough we refer here to a conceptual map for orienting the reader along this pattern.

2.3. Chain Ladder Method: Basic Concept

The idea underlying the chain ladder method is that there is a proportion between the cumulative payments of two close development years, except for an erratic component with a null mean:

$$C^I_{i,j+1} = C_{i,j} f_j + \xi_{ij}, \qquad i = 0, 1, \ldots, I - j - 1, \tag{3}$$

looking at Equation (3), we conclude that, in the chain ladder model, the cumulative payments are showed by a line through the origin for each j development year. If we assume the residuals variance is $Var(\xi_{ij}) = \sigma^2 C_{i,j}$, the least square solution for the f_j estimate is:

$$\hat{f}^I_j = \frac{\sum_{k=0}^{I-j-1} C_{k,j+1}}{\sum_{k=0}^{I-j-1} C_{k,j}} = \frac{\sum_{k=0}^{I-j-1} C_{k,j} F_{k,j}}{\sum_{k=0}^{I-j-1} C_{k,j}}, \qquad j = 0, 1, \ldots, J - 1, \tag{4}$$

which is the weighted average of all link ratios observed. This approach implies that the cumulative payments $C_{i1,j}$ and $C_{i2,j}$ for $i_1 \neq i_2$ are independent; each ratio j, beyond being independent from the i generation, must also have equal first two moments with a fixed j, thus the process of claim settlement must not have undergone structural changes in time. The ultimate cost $\hat{C}^I_{i,J}$ estimate is calculated through the use of the factors $\hat{f}^I_j$:

$$\hat{C}^I_{i,J} = C_{i,I-i} \prod_{j=I-i}^{J-1} \hat{f}^I_j, \quad i = 1, \ldots, I, \tag{5}$$

thus the claims reserve estimate is:

$$\hat{R}^I_i = \hat{C}^I_{i,J} - C_{i,I-i}, \quad i = 1, \ldots, I. \tag{6}$$

2.4. Chain Ladder Method: time series approach

For each accident year, the chain ladder model can be viewed as an autoregressive model where the innovations are not white noise but they are heteroskedastic and dependent on the square-root of the previous observation. In [Merz and Wüthrich(2008b)] the time series hypothesis has changed by three assumptions: mean, variance and Markov process. Nevertheless all the results have based on times series approach. We assume that exist costants f_j and σ_j such that:

$$C_{i,j} = f_{j-1} C_{i,j-1} + \sigma_{j-1} \sqrt{C_{i,j-1}}\, \varepsilon_{i,j}, \quad i = 1, \ldots, I \quad j = 1, \ldots, J - 1 \tag{7}$$

where $C_{i,0} > 0$, $E\,[\varepsilon_{i,j}]\,l = 0$ and $E[\varepsilon^2_{i,j}] = 1$, therefore the mean and Variance of cumulative payments are $E[C_{i,j} \mid C_{t,j-1}] = f_{j-1} C_{i,j-1}$ and $Var\ [C_{i,j} \mid C_{i,j-1}] = \sigma^2_{j-1} C_{i,j-1}$ respectively.

These assumptions have to be verified for data fit with graphical analyses to highlight the accident years independence and the residuals distribution and with a regression analysis to evaluate the materiality of development factors. The same assumptions, using the tower property of conditional expectations, allow to define the expected ultimate cost as:

$$E[C_{i,J} \mid D_I] = C_{i,I-i} \prod_{j=I-i}^{J-1} f_j, \qquad i = 1,\ldots,I \tag{8}$$

where $D_I = \{C_{i,j};\ i + j \le I \quad i \le I\}$ denote the claims data available at time $t = I$. In the same way at time $t = I + 1$ the expected vale is $E \mid C_{i,J} \mid D_{I+1}] = C_{i,I-i+1} \prod_{j=I-i+1}^{J-1} f_j$. Wherever the development factors are known, we are able to calculate the unconditional expected ultimate costs provided the available information. However these factors are unknown and have to be estimated as ratio between sums of cumulative payments at different development moments through (4) that we rewrite in the convenient form:

$$\begin{aligned} \hat{f}_j^I &= \frac{\Sigma_{i=0}^{I-j-i} C_{i,j+1}}{S_j^I} \ \ with\ \ S_j^I = \Sigma_{i=0}^{I-j-1} C_{i,j} \\ \hat{f}_j^{I+1} &= \frac{\Sigma_{i=0}^{I-j} C_{i,j+1}}{S_j^{I+1}} \ \ with\ \ S_j^{I+1} = \Sigma_{i=0}^{I-j} C_{i,j}. \end{aligned} \tag{9}$$

Estimates quantified in the next year are enhanced by incremental information on claims development result, within the two instants. [Mack(1993)] demonstrated that estimators are unbiased and uncorrelated, for development year, thus the estimators for the expected ultimate cost are:

$$\begin{aligned} E \mid C_{i,J} \mid D_I] &= \hat{C}_{i,J}^I &&= \underbrace{C_{i,I-i} \hat{f}_{I-i}^I}_{\hat{C}_{i,I-i+1}^I} \ \hat{f}_{I-i+1}^I \cdots \hat{f}_{J-2}^I \hat{f}_{J-1}^I \\ E \mid C_{i,J} \mid D_{I+1}] &= \hat{C}_{i,J}^{I+1} &&= C_{i,I-i+1} \ \ \hat{f}_{I-i+1}^{I+1} \cdots \hat{f}_{J-2}^{I+1} \hat{f}_{J-1}^{I+1}. \end{aligned} \tag{10}$$

2.5. Merz,Wütrich and Lysenko's Lemma

As introduced in [Wüthrich *et al.*(2008)] and under the model assumptions 2.4, we have the following the Merz,Wütrich and Lysenko's Lemma that it will be useful for further elaboration:

a) $C_{i,I-i+1}, \hat{f}^{I+1}_{I-i+1}, \cdots, \hat{f}^{I+1}_{J-1}$ are independent conditionally to D_I;

b) Developmpent factor's expectation:

$$E\left[\hat{f}^{I+1}_j \mid D_I\right] = \hat{f}^I_j \frac{s^I_j}{s^{I+1}_j} + f_j \frac{C_{I-j.j}}{s^{I+1}_j},$$

c) Conditional mean at time I of the estimate ultimate cost at time $I+1$ $E\left[\hat{C}^{I+1}_{i,J} \mid D_I\right] = C_{i,I-i} f_{I-i} \prod_{j=I-i+1}^{J-1} E\left[\hat{f}^{I+1}_j \mid D_I\right]$;

d) Conditional squared mean at time I of the cumulative payment at time $I+1$

$$E\left[C^2_{i,I-i+1}\right] = f^2_{I-i} C^2_{i,I-i} + \sigma^2_{I-i} C_{i,I-i};$$

e) Conditional squared mean at time I of the development factor at time $I+1$

$$E\left[\left(\hat{f}^{I+1}_j\right)^2 \mid D_I\right] = \left(\hat{f}^{I+1}_j \frac{\Sigma_{i=0}^{I-j} C_{i,j+1}}{s^{I+1}_j} + f_j \frac{C_{I-j,j}}{S^{I+1}_j}\right)^2 + \frac{\sigma^2_j C_{I-i,i}}{\left(S^{I+1}_j\right)^2};$$

f) Conditional product mean at time I of the development factor at time $I+1$

$$E\left[C_{i,I-i+1} \hat{f}^{I+1}_j \mid D_I\right] = \frac{1}{S^{I+1}_{I-i}} \left(f^2_{I-i} C^2_{I-i} + \sigma^2_{I-i} C_{I-i} + S^{I+1}_{I-i} f_{I-i} C_{I-i}\right).$$

Proof of the Lemma:

a)

$$\begin{aligned} C_{i,I-i+1} &= f_{I-i} C_{i,I-i} + \sigma_{I-i} \sqrt{C_{i,I-i} \varepsilon_{i,I-i+1}} \quad \varepsilon_{i,I-i+1} \\ f^{I+1}_j &= \frac{\Sigma_{i=0}^{I-j} C_{i,j+1}}{s^{I+1}_j} = \frac{\Sigma_{i=0}^{I-j-1} C_{i,j+1}}{s^{I+1}_j} + \frac{C_{I-j,j+1}}{s^{I+1}_j} = \hat{f}^I_j \frac{s^I_j}{s^{I+1}_j} + \frac{f_{j-1} C_{i,j-1} + \sigma_{j-1} \sqrt{C_{i,j-1}}}{s^{I+1}_j} \end{aligned}$$

is function of the r.v. $\varepsilon_{I-j,j+1}$ that it is in the range $(\varepsilon_{i-1,I-i+2}, \varepsilon_{I-J+1,J})$. Since for the model assumptions the residuals are independent the a) is proved;

$$E\left[\hat{f}_j^{I+1} \mid D_I\right] = \frac{\Sigma_{i=0}^{I-j-1} C_{i,j+1}}{s_j^{I+1}} + \frac{E[C_{I-j,j+1} \mid D_I]}{s_j^{I+1}} = \frac{\Sigma_{i=0}^{I-j-i} C_{i,j+1}}{s_j^{I+1}} + f_j \frac{C_{I-j,j}}{s_j^{I+1}} = \hat{f}_j^I$$

b) $$\frac{s_j^I}{s_j^{I+1}} + f_j \frac{C_{I-j,j}}{s_j^{I+1}},$$

e)

$$E\left[\left(\hat{f}_j^{I+1}\right)^2 \mid D_I\right] =$$

$$= E\left[\left(\frac{\Sigma_{i=0}^{I-j-1} C_{i,j+1}}{s_j^{I+1}} + \frac{C_{I-j,j+1}}{s_j^{I+1}}\right)^2 \mid D_I\right] =$$

$$= \left(\frac{\Sigma_{i=0}^{I-j-1} C_{i,j+1}}{s_j^{I+1}}\right)^2 + \frac{E[C_{I-j,j+1}^2 \mid D_I]}{\left(s_j^{I+1}\right)^2} + 2\frac{\Sigma_{i=0}^{I-j-1} C_{i,j+1}}{\left(s_j^{I+1}\right)^2} E[C_{I-j,j+1} \mid D_I] =$$

$$= \left(\frac{\Sigma_{i=0}^{I-j-1} C_{i,j+1}}{s_j^{I+1}}\right)^2 + \frac{f_j^2 C_{I-j,j}^2 + \sigma_j^2 C_{I-j,j}}{\left(s_j^{I+1}\right)^2} + 2\frac{\Sigma_{i=0}^{I-j-1} C_{i,j+1}}{\left(s_j^{I+1}\right)^2} f_j C_{I-j,j} =$$

$$= \left(\frac{\Sigma_{i=0}^{I-j-1} C_{i,j+1}}{s_j^{I+1}} + f_j \frac{C_{I-j,j}}{s_j^{I+1}}\right)^2 + \frac{\sigma_j^2 C_{I-j,j}}{\left(s_j^{I+1}\right)^2};$$

f) $$E\left[C_{i,I-i+1} \hat{f}_j^{I+1} \mid D_I\right] =$$

$$= \frac{1}{s_{I-i}^{I+1}} E[C_{i,I-i+1} \Sigma_{k=0}^{i} C_{k,I-i+1} \mid D_I] =$$

$$= \frac{1}{s_{I-i}^{I+1}} E\left[C_{i,I-i+1}^2 \mid D_I\right] + \frac{1}{s_{I-i}^{I+1}} \Sigma_{k=0}^{i} C_{k,I-i+1} E[C_{i,I-i+1} \mid D_I] =$$

$$= \frac{1}{s_{I-i}^{I+1}} \left(f_{I-i}^2 C_{I-i}^2 + \sigma_{I-i}^2 C_{I-i} + S_{I-i}^{I+1} f_{I-i} C_{I-i}\right).$$

In several steps we will use the following approximation:

$$\prod_{j=1}^{J}(1+a_j) \approx \sum_{j=1}^{J} a_j \quad with \qquad a_j >> 1 \tag{11}$$

3. MACK MODEL REVISED

In this section it is revised the model proposed in [Mack(1993)], which it is widely used to estimate the mean square error of prediction(MSEP) of the ultimate cost in the chain ladder framework:

$$MSEP_{C_{i,J}|D_I} = E\left[\left(C_{i,J} - \hat{C}_{i,J}\right)^2 | D_I\right] = \underbrace{Var(C_{i,J} | D_I)}_{process} + \underbrace{\left(E[C_{i,J} | D_I] - \hat{C}_{i,J}\right)^2}_{parameter} \tag{12}$$

the MSEP can be divided into *process* and *parameter* error, the first one is related to variance of r.v. $C_{i,J}$ while the second is linked to the bias of the estimator $\hat{C}_{i,J}$.

3.1. Estimation of process error for a single accident year

The variance of process error can be write through the law of total variance:

$$\begin{aligned}\text{Var}\,[C_{i,J} \,|\, D_I] &= \text{Var}\,[C_{i,J} \,|\, C_{i,I-i}] = E\,[\text{Var}\,[C_{i,J} \,|\, C_{i,J-1}] \,|\, C_{i,I-i}] + \text{Var}\,[E\,[C_{i,J} \,|\, C_{i,J-1}] \\ &\quad |\, C_{i,I-i}] \\ &= \sigma_{J-1}^2\, E[C_{i,J-1} | C_{I-i}] + f_{J-1}^2\, Var[C_{i,J-1} | C_{i,I-i}] \\ &= \sigma_{J-1}^2\, C_{i,I-i} \textstyle\prod_{l=I-i}^{J-2} f_l + f_{J-I}^2\, Var[C_{i,J-1} | C_{i,I-i}]\end{aligned} \tag{13}$$

by means of (13) a recursion procedure can be start and process variance becomes:

$$Var[C_{i,J} | C_{i,I-i}] = E[C_{i,J} | C_{i,I-i}]^2 \sum_{j=1-i}^{J-1} \frac{\beta_j^2}{E[C_{j,J} | C_{i,I-i}]}, \quad i = 1,\ldots,I, \tag{14}$$

where $\beta_j = \sigma_j / f_j$ is a coefficient of variation.

Using is the Mack's estimator for σ_j:

$$\begin{aligned} \hat{\sigma}_j^2 &= \frac{1}{I-j-1} \Sigma_{i=0}^{I-j-1} C_{ij} \left(\frac{C_{i,j+1}}{C_{i,j}} - \hat{f}_j \right)^2 & j = 0, \ldots, J-2 \\ \hat{\sigma}_{J-1}^2 &= \min \left(\hat{\sigma}_{J-2}^4 / \hat{\sigma}_{J-3}^2, \hat{\sigma}_{J-3}^2, \hat{\sigma}_{J-2}^2 \right) & j = J-1 \end{aligned} \tag{15}$$

the estimate of process error is:

$$V\hat{a}r[C_{i,J} \mid D_I] = \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\beta}_j^2}{\hat{C}_{j,J}}, \quad i = 1, \ldots, I. \tag{16}$$

3.2. Estimation of parameter error for a single accident year

In order to calculate the parameter error for a specific accident year, it is necessary to determine the fluctuation of the square estimators of development factors $f_{I-i}^2, \ldots, f_{J-1}^2$ around the true values $f_{I-i}^2, \ldots, f_{J-1}^2$. To realize this the estimators volatility has to be specified through conditional re-sampling technique: provided information in I we generate new observations:

$$Z_{i,j} = f_{j-1} C_{i,j-1} + \sigma_{j-1} \sqrt{C_{i,j-1}} \varepsilon_{i,j} \tag{17}$$

that lead to new realizations for the expected estimated development factors

$$\hat{f}_j^I = \frac{\Sigma_{i=0}^{I-j-1} Z_{i,j+1}}{\Sigma_{i=0}^{I-j-1}} = f_j + \frac{\sigma_j}{S_j^I} \sum_{i=0}^{I-j-1} \sqrt{C_{i,j}} \varepsilon_{i,j+1} \qquad 0 \le j \le J-1, \tag{18}$$

differently from the observed cumulative payments, the new observations of development factors $\hat{f}_j^I$ are random variables; moreover the initial observations are unconditionally independent from $\varepsilon_{i,j}$ and $\hat{f}_0^I, \ldots, \hat{f}_{J-1}^I$ are conditionally independent on D_I, so the expected value and variance are:

$$E\left[\hat{f}_{j-1}^I \mid D_I \right] = f_{j-1} \text{ and } Var\left[\hat{f}_{j-1}^I \mid D_I \right] = \frac{\sigma_{j-1}^2}{S_{j-1}^I} \tag{19}$$

Now using the results above we are able to obtain the expected value of *parameter error* in 12:

$$E\left[(E[C_{i,J} \mid D_I] - \hat{C}_{i,J})^2 \mid D_I\right] =$$

$$= E\left[C_{i,I-i}^2\left(\Pi_{j=I-i}^{J-1}\left(\hat{f}_j^I\right)^2 + \Pi_{j=I-i}^{J-1} f_j^2 - 2\Pi_{j=I}^{J-1} \hat{f}_j^I\, \Pi_{j=I}^{J-1} f_j\right) \mid D_I\right] =$$

$$= C_{i,I-i}^2\left(\Pi_{j=I-i}^{J-1} E\left[\left(\hat{f}_j^I \mid D_I\right)^2\right] + \Pi_{j=I-i}^{J-1} f_j^2 - 2\Pi_{j=I-i}^{J-1} E\left[\hat{f}_j^I \mid D_I\right]\Pi_{j=I}^{J-1} f_j\right)$$

$$= C_{i,I-i}^2\left(\Pi_{j=I-i}^{J-1}\left(\frac{\sigma_j^2}{s_j^I} + f_j\right) + \Pi_{j=I-i}^{J-1} f_j^2 - 2\Pi_{j=I-i}^{J-1} f_j \Pi_{j=I-i}^{J-1} f_j\right)$$

$$= C_{i,I-i}^2\left(\Pi_{j=I-i}^{J-1}\left(\frac{\sigma_j^2}{s_j^I} + f_j\right) - \Pi_{j=I-i}^{J-1} f_j^2\right)$$

$$= C_{i,I-i}^2 \Pi_{j=I-i}^{J-1} f_j^2\left(\Pi_{j=I-i}^{J-1}\left(\frac{\sigma_j^2 / f_j^2}{s_j^I} + 1\right) - 1\right)$$

$$\approx C_{i,I-i}^2 \Pi_{j=I-i}^{J-1} f_j^2 \Sigma_{j=I-i}^{J-1} \frac{\sigma_j^2 / f_j^2}{s_j^I}$$

$$= C_{i,I-i}^2\, \Pi_{j=I}^{J-1} f_j^2\, \Sigma_{j=I-i}^{J-1}\, \eta_j^2,$$

where $\eta_j^2 = \dfrac{\sigma_j^2 / f_j^2}{s_j^I}$ is the coefficient of variation normalized to the square root basis for calculation of the factors at time I.

Substituting with the estimators we get the estimate of parameter error for a single accident year:

$$\hat{E}\left[(E[C_{i,J} \mid D_I] - \hat{C}_{i,J})^2 \mid D_I\right] \approx \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2 / \hat{f}_j^2}{S_j^I} = \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \hat{\eta}_j^2. \tag{21}$$

3.3. Estimation of Mean Square Error of Prediction

The mean square error of prediction for a **single accident year** in (12) can be estimated substituting the estimator in (16) and (21), so we get the following estimate:

$$M\hat{S}EP_{\hat{C}_{i,J}}(\hat{C}_{i,J}) = \hat{C}_{i,J}^2 \sum_{j=I-i}^{J-1} \left(\frac{\hat{\beta}_j^2}{\hat{C}_{i,j}} + \hat{\eta}_j \right). \tag{22}$$

Table 2
Cumulative payment used in the empirical application (,000)

i/j	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*
0	22,603	62,541	97,614	123,163	143,194	160,787	175,717	190,721	201,040	209,280	217,384	223,404	242,549
1	22,382	63,884	90,392	110,126	128,841	142,824	155,709	172,080	180,001	187,205	191,633	204,530	
2	25,355	71,062	104,124	128,356	145,121	158,301	169,940	178,804	188,798	194,842	198,796		
3	26,830	79,177	116,501	140,091	158,339	172,234	185,376	196,495	205,924	210,981			
4	26,868	89,181	122,953	145,878	162,219	174,638	187,284	196,743	203,401				
5	28,470	84,567	126,239	151,082	173,900	192,687	209,634	224,576					
6	26,170	81,532	120,558	147,375	170,256	189,919	209,314						
7	24,101	82,621	121,370	143,819	159,827	172,333							
8	22,714	71,421	100,391	119,189	132,558								
9	19,973	58,235	81,533	96,352									
10	17,252	54,246	78,607										
11	17,591	47,665											
12	16,907												

In order to illustrate the numerical computation we use the cumulative payment C_{ij} in Table 2 that it is obtained by the triangle of incremental payments used in [Cavastracci and Tripodi(2018)]

Listing 1: Code for Table 3: Estimation f, σ, β, η.

```
## Cumulative.Paid is the run-off triangle in Table 2.
## It must be as.matrix and future value are indicated as NA.

n.origin      <-  nrow(Cumulative.Paid)
n.dev         <-  ncol(Cumulative.Paid)
origin        <-  0:(n.origin-1)
dev           <-  0:(n.dev-1)
dev1          <-  0:(n.dev-2)

last.diag.index     <- n.origin+(n.origin-1)*(0:(n.origin-1))
Last.Diag           <- cbind(0:(n.dev-1),(n.origin-1):0,Cumulative.
                       Paid[last.diag.index])
colnames(Last.Diag)<- c('Dev', 'Origin', 'Cij')

## equation (9), estimates f_{j}^{I}
S_I1          <-  apply(Cumulative.Paid,2,sum,na.rm=T)
S_I           <-  S_I1 - Last.Diag[,'Cij']
```

```
names(S_I)    <- names(S_I1) <- dev
f             <-  S_I1[-1]/S_I[-n.origin]
names(f)      <- dev1

## equation (10), estimates C_{ij}^{I}
hat_Cij                    <-matrix(,n.origin,n.dev)
hat_Cij[last.diag.index]  <-Last.Diag[,'Cij']

for(i in 2:n.origin){
       hat_Cij[i,(n.origin-i+2):n.dev] <-
       Last.Diag[n.origin-i+1,'Cij']*cumprod(f[(n.origin-i+1):
       (n.origin-1)])
}
hat_CiJ <- hat_Cij[,n.dev]
names(hat_CiJ) <- dev
## individual link ratio
Fij <- Cumulative.Paid[-n.origin,2:(n.origin-1)]/
       Cumulative.Paid[-n.origin,1:(n.origin-2)]
## intermediate calculation
Fij_sum <- apply(Cumulative.Paid[-n.origin,1:(n.origin-2)]*(Fij^2),
2,sum,na.rm=T)
## equation (14)
sigma_temp <-
sqrt( (Fij_sum -
(f[-(n.origin-1)]^2)*S_I[-c(n.origin-1,n.origin)]) /
((n.origin-2):1))

sigma <-c(sigma_temp,
sqrt(min(tail(sigma_temp,1)^4/tail(sigma_temp,2)^2,
tail(sigma_temp,2)^2,tail(sigma_temp,1)^2)))
##
names(sigma) <-dev1
beta <- sigma/f
beta2 <- beta^2
eta2 <-(beta^2)/S_I[dev1+1]
eta <-sqrt(eta2)
Table 3 <- cbind(f,sigma,beta,eta)
```

Table 3
Estimation of Mack's parameters using the run-off data in Table 2. The results obtain running the code 1

j	$\hat{f}_j^I$	$\hat{\sigma}_j$	$\hat{\beta}_j$	$\hat{\eta}_j$
0	3.0186	33.9052	11.2322	0.0212
1	1.4531	13.7190	9.4409	0.0106
2	1.2069	8.2941	6.8725	0.0066
3	1.1366	8.2933	7.2965	0.0066
4	1.0983	6.9007	6.2832	0.0056
5	1.0853	4.7015	4.3321	0.004
6	1.0699	8.6955	8.1273	0.0078
7	1.0474	3.809	3.6366	0.0038
8	1.0342	3.4356	3.3219	0.0038
9	1.0279	4.4252	4.3052	0.0056
10	1.0462	12.6403	12.0815	0.0189
11	1.0857	4.4252	4.0759	0.0086

In order to obtain the estimate of mean square error of prediction for all accident years, we consider at first the estimation for two accident years k and i (with $k > i$):

$$
\begin{aligned}
&MSEP_{C_{iJ}+C_{kJ}}\left(\hat{C}_{iJ}^I+\hat{C}_{kJ}^I\right)=E\left[\left(\hat{C}_{iJ}^I+\hat{C}_{kJ}^I-(C_{iJ}+C_{kJ})\right)^2 \mid D_I\right]=\\
&=\underbrace{Var(C_{iJ}+C_{kJ}\mid D_I)}_{process}+\underbrace{\left(\hat{C}_{iJ}^I+\hat{C}_{kJ}^I-E[C_{iJ}+C_{kJ}\mid D_I]\right)^2}_{parameter}=\\
&=\underbrace{Var(C_{iJ}\mid D_I)+Var(C_{kJ}\mid D_I)}_{process}+\\
&+\underbrace{\left(\hat{C}_{iJ}^I-E[C_{iJ}\mid D_I]\right)^2+\left(\hat{C}_{kJ}^I-E[C_{kJ}\mid D_I]\right)^2+2\left(\hat{C}_{iJ}^I-E[C_{iJ}\mid D_I]\right)\left(\hat{C}_{kJ}^I-E[C_{kJ}\mid D_I]\right)}_{parameter}=\\
&=E\left[\left(\hat{C}_{iJ}-C_{iJ}\right)^2\mid D_I\right]+E\left[\left(\hat{C}_{kJ}-C_{kJ}\right)^2\mid D_I\right]+2\left(\hat{C}_{iJ}^I-E[C_{iJ}\mid D_I]\right)\left(\hat{C}_{kJ}^I-E[C_{kJ}\mid D_I]\right)\\
&=MSEP_{C_{iJ}\mid D_I}\left(\hat{C}_{iJ}^I\right)+MSEP_{C_{kJ}\mid D_I}\left(\hat{C}_{kJ}^I\right)+2\left(\hat{C}_{iJ}^I-E[C_{iJ}\mid D_I]\right)\left(\hat{C}_{kJ}^I-E[C_{kJ}\mid D_I]\right),
\end{aligned}
\tag{23}
$$

from (23) it results that the MSEP for two accident year is equal to the sum of MSEPs plus the blue formula that it derives from the correlation between the estimators $\hat{C}_{iJ}$ and $\hat{C}_{kJ}$, it is the parameter error between two accident years. The last one can be estimated using the results of conditional re-sampling technique:

$$E[(\hat{C}^I_{iJ} - E[C_{iJ} \mid D_I])\,(\hat{C}^I_{kJ} - E[C_{kJ} \mid D_I]) \mid D_I] =$$

$$= E\left[\left(C_{i,I-i}\left(\Pi_{j=I-i}^{J-1}\hat{f}_j^I - \Pi_{j=I-i}^{J-1} f_j\right)\right)\left(C_{k,I-k}\left(\Pi_{j=I-k}^{J-1}\hat{f}_j^I - \Pi_{j=I-k}^{J-1} f_j\right)\right) \mid D_I\right] =$$

$$C_{i,I-i}C_{k,I-k}E\left[\Pi_{j=I-i}^{J-1}\hat{f}_j^I\Pi_{j=I-k}^{J-1}\hat{f}_j^I - \Pi_{j=I-i}^{J-1} f_j \Pi_{j=I-k}^{J-1}\hat{f}_j^I - \Pi_{j=I-i}^{J-1}\hat{f}_j^I\Pi_{j=I-k}^{J-1} f_j +\right]$$

$$+ \Pi_{j=I-i}^{J-1} f_j\, \Pi_{j=I-k}^{J-1} f_j \mid D_I] =$$

$$= C_{i,I-i}C_{k,I-k}E\left[\Pi_{j=I-k}^{I-i-1}\hat{f}_j^I\left(\Pi_{j=I-i}^{J-1}\left(\hat{f}_j^I\right)^2 - \Pi_{j=I-i}^{J-1} f_j\Pi_{j=I-i}^{J-1}\hat{f}_j^I\right) + \right.$$

$$\left. + \Pi_{j=I-k}^{I-i-1} f_j^I\left(\Pi_{j=I-i}^{J-1} f_j^2 - \Pi_{j=I-i}^{J-1} f_j\Pi_{j=I-i}^{J-1}\hat{f}_j^I\right) \mid D_I\right] =$$

$$= C_{i,I-i}C_{k,I-k}\Pi_{j=I-k}^{I-i-1}E\left[\hat{f}_j^I \mid D_I\right]\left(\Pi_{j=I-i}^{J-1}E\left[\left(\hat{f}_j^I\right)^2 \mid D_I\right] - \Pi_{j=I-i}^{J-1} f_j\Pi_{j=I-i}^{J-1}E\left[\hat{f}_j^I \mid D_I\right]\right) =$$

$$= C_{i,I-i}C_{k,I-k}\Pi_{j=I-k}^{I-i-1} f_j\left(\Pi_{j=I-i}^{J-1}E\left[\left(\hat{f}_j^I\right)^2 \mid D_I\right] - \Pi_{j=I-i}^{J-1} f_j^2\right) =$$

$$= C_{i,I-i}C_{k,I-k}\Pi_{j=I-k}^{I-i-1} f_j\left(\Pi_{j=I-i}^{J-1}\left(f_j^2 + \frac{\sigma_j^2}{s_j^I}\right) - \Pi_{j=I-i}^{J-1} f_j^2\right) =$$

$$= C_{i,I-i}C_{k,I-k}\Pi_{j=I-k}^{I-i-1} f_j\Pi_{j=I-i}^{J-1} f_j^2\left(\Pi_{j=I-i}^{J-1}\left(1 + \frac{\sigma_j^2 / f_j^2}{s_j^I}\right) - 1\right) \approx$$

$$\approx C_{i,I-i}C_{k,I-k}\Pi_{j=I-k}^{I-i-1} f_j\Pi_{j=I-i}^{J-1} f_j^2\Sigma_{j=I-i}^{J-1}\frac{\sigma_j^2 / f_j^2}{s_j^I} =$$

$$= C_{i,I-i}C_{k,I-k}\Pi_{j=I-k}^{I-i-1} f_j\Pi_{j=I-i}^{J-1} f_j^2\Sigma_{j=I-i}^{J-1}\eta_j^2, \qquad (24)$$

substituting the unknown f_j and η_j with them estimators $\hat{f}_j$ and $\hat{\eta}_j$ then we get the corresponding estimate:

$$\hat{E}\left[\left(\hat{C}_{iJ}^{I} - E[C_{iJ} \mid D_I]\right)\left(\hat{C}_{kJ}^{I} - E[C_{kJ} \mid D_I]\right)\right] = \hat{C}_{iJ}^{I}\,\hat{C}_{kJ}^{I} \sum_{j=I-i}^{J-1} \hat{\eta}_j^2. \tag{25}$$

Finally the estimate of the overall mean square error of prediction that it is the Mack's formula in [Mack(1993)]:

$$M\hat{S}EP_{\Sigma_{i=1}^{I} C_{iJ} \mid D_I}\left(\sum_{i=1}^{J} \hat{C}_{iJ}^{I}\right) = \sum_{i=1}^{J} M\hat{S}EP_{C_{iJ} \mid D_I}\left(\hat{C}_{iJ}^{I}\right) + 2\sum_{k>i} \hat{C}_{iJ}^{I}\hat{C}_{kJ}^{I} \sum_{j=I-i}^{J-1} \hat{\eta}_j^2 \tag{26}$$

Listing 2: Code for Table 4: Estimation MSEP for ultimate cost.

```
#The Code 1 must be runned before this.

## equation(6) claims reserve estimate
hat_R_i      <- hat_CiJ - Last.Diag[n.origin:1, 'Cij']
##
hat_Cij_rec <- 1/hat_Cij
hat_Cij_rec[which(is.na(hat_Cij_rec))] <- 0

## equation (15), estimation of process error for ultimate cost.
hat_Process.Error_CiJ <- (hat_CiJ^2)*
(hat_Cij_rec[1:n.origin, dev1+1] %*% beta2)

## equation (20), estimation of process error for ultimate cost.
hat_Estimation.Error_CiJ <-(hat_CiJ^2)*c(0,cumsum(eta2[(n.origin-
1):1]))

## equation (21) = equation (15) + equation (20), estimation of MSEP
for ultimate cost
MSEP_hatCiJ  <- hat_Process.Error_CiJ+hat_Estimation.Error_CiJ
names(MSEP_hatCiJ) <- origin

## equation (25), estimation MSEP for overall generation
covariance <- 2*sum(hat_CiJ[2:n.origin]*
c(cumsum(hat_CiJ[n.origin:3])[(n.origin-2):1],0)*
cumsum(eta2[(n.origin-1):1]))
MSEP_MACK_tot <- sum(MSEP_hatCiJ)+covariance
```

```
Table.4 <- rbind(cbind(hat_R_i,hat_CiJ, sqrt(hat_Process.Error_CiJ),
sqrt(hat_Estimation.Error_CiJ),  sqrt(MSEP_hatCiJ)),

  c(sum(hat_R_i),sum(hat_CiJ),NA,NA,sqrt(MSEP_MACK_tot)))
colnames(Table.4) <- c('hat.R.i ', 'hat.CiJ ', 'r.Process.Error',
'r.Estimation.Error ', 'rMSEP ')
```

Table 4
Estimation of rMSEP with the Mack model. The output is made by the code 2

i	$\hat{R}_i$	$\hat{C}_{iJ}$	$\sqrt{\widehat{VAR} \mid C_{i,J} \mid D_I}$	$\sqrt{\hat{E}\left[(E[C_{i,J} \mid D_I] - \hat{C}_{i,J})^2 \mid D_I\right]}$	$\sqrt{\widehat{MSEP}_{\hat{C}_{i,J}}\left(\hat{C}_{i,J}\right)}$
0	0	242,549	0	0	0
1	17,528	222,058	2,001	1,915	2,770
2	27,018	225,814	6,443	4,689	7,969
3	35,356	246,337	7,115	5,298	8,871
4	42,212	245,613	7,331	5,363	9,083
5	59,463	284,039	8,179	6,294	10,320
6	73,930	283,244	9,593	6,654	11,675
7	80,752	253,085	9,445	6,030	11,206
8	81,245	213,803	9,433	5,235	10,788
9	80,285	176,637	9,526	4,481	10,527
10	95,309	173,916	10,369	4,559	11,327
11	105,579	153,244	11,775	4,331	12,547
12	147,172	164,079	18,691	5,798	19,570
Tot.	845,851	2,884,420	-	-	65,183

"hat.CiJ","r.Process.Error","r.Estimation.Error","rMSEP")

4. THE ONE-YEAR VOLATILITY

The one-year volatility is estimated by means of the CDR's definition like in section 2.2. In this section we define the CDR as the difference of conditional expectation between two consecutive balance years:

$$\begin{aligned} CDR_i(I+1) \quad &= E[R_i^I \mid D_I] - (X_{i,I-i+1} + E[R_i^{I+1} \mid D_{I+1}]) = \\ &= E[C_{i,J} \mid D_I] - E[C_{i,J} \mid D_{I+1}], \quad i = 1, \ldots, I \end{aligned} \tag{27}$$

the stochastic process $CDR_i(I+1)$ is a martingala thus $E\,[CDRi(I+1) \mid DI\,] = 0$, namely the expected run off is zero at time I.

The variance of CDR for a single accident year can be calculated as:

$$\begin{aligned}
\text{Var}\,[CDR_i\,(I+1)\ \,|\,D_I] &= \text{Var}\,[E\,[C_{i,J}\ |\,D_I] - E\,[C_{i,J}\ |\,D_{I+1}]] = \\
&= \text{Var}\,[E\,[C_{i,J}\ |\,D_{I+1}]] = \\
&= \text{Var}\,[C_{i,I-i+1}\,|\,D_I]\ \Pi_{j=I-i+1}^{J-1} f_j^2 = \qquad (28)\\
&= \sigma_{I-i}^2 C_{i,I-1} \Pi_{j=I-i+1}^{J-1} f_j^2 = \\
&= E[C_{iJ} \mid D_I]^2 \frac{\sigma_{I-i}^2 / f_{I-i}^2}{C_{i,I-i}} = \\
&= E[C_{iJ} \mid D_I]^2\ \theta_{I-i}^2, \qquad i = 1,\ldots,I,
\end{aligned}$$

where $\theta_{I-i}^2 = \frac{\sigma_{I-i}^2 / f_{I-i}^2}{C_{i,I-i}}$ is the coefficient of variation normalized to the square root basis for calculation of the individual factor.

Since the payments among accident years are independent, then the variance for overall accident years is calculated as the sum of variance of single years:

$$Var\left[\sum_{i=1}^{I} CDR_i(I+1) \mid D_I\right] = \sum_{i=1}^{I} E[C_{iJ} \mid D_I]^2\, \theta_{I-i}^2. \qquad (29)$$

The estimator for *CDR* is:

$$\overline{CDR_i}(I+1) = \hat{R}_i^I - \left(X_{i,I-i+1} - \hat{R}_i^{I+1}\right) = \hat{C}_{iJ}^I - \hat{C}_{iJ}^{I+1} \qquad (30)$$

the estimator for overall *CDR* is computed as the sum of the single estimators.

In a *retrospective view* we have the mean square error of prediction of the observed *CDR* that it measures the quality of approximation:

$$MSEP_{\overline{CDR_i}(I+1)|D_I}\left(\overline{CDR_i}(I+1)\right) = E\left[\left(CDR_i(I+1) - \overline{CDR_i}(I+1)\right)^2 \mid D_I\right], \qquad (31)$$

while in a *prospective view* (as required in Solvency II) the mean square error of prediction of the *CDR* is computed from zero, it measures the quality of prediction:

$$MSEP_{\overline{CDR_i}(I+1)D_I}(0) = E\left[(CDR_i(I+1)-0)^2 \mid D_I\right]. \tag{32}$$

4.1. Mean square error of prediction for a single accident year

In order to estimate the CDR in a retrospective view, the (31) can be written as:

$$\begin{aligned}
&MSEP_{\overline{CDR_i}(I+1)D_I}\left(\overline{CDR_i}(I+1)\right) = \\
&= \underbrace{Var\left[\left(CDR_i(I+1)-\overline{CDR_i}(I+1)\right) \mid D_I\right]}_{process} + \underbrace{E\left[\left(CDR_i(I+1)-\overline{CDR_i}(I+1)\right) \mid D_I\right]^2}_{parameter} \\
&= \underbrace{\varphi_{iJ}^I}_{process} + \underbrace{E\left[\overline{CDR_i}(I+1)\right]^2}_{parameter}
\end{aligned}$$

The *process error* φ_{iJ}^I for the first accident year ($i = 1$) is null, this result is natural since we consider that all claims has been paid in J years, thus $\varphi_{iJ}^I = 0$. For $i > 1$ the *process error* can be decomposed using the properties a) e d) of the Lemma 2.5:

$$\begin{aligned}
\varphi_{iJ}^I &= Var[CDR_i(I+1) \mid D_I] + Var\left[\overline{CDR_i}(I+1) \mid D_I\right] - 2Cov\left[CDR_i(I+1), \overline{CDR_i}(I+1)\right] = \\
&= E[C_{iJ} \mid D_I]^2 \, \theta_{I-i}^2 + Var\left(\hat{C}_{iJ}^{I+1}\right) - 2Cov\left[E[C_{iJ} \mid D_I], \hat{C}_{iJ}^{I+1}\right] = \\
&= E[C_{iJ} \mid D_I]^2 \, \theta_{I-i}^2 + \\
&+ E\left[\left(\hat{C}_{iJ}\right)^2 \mid D_I\right] - E[C_{iJ} \mid D_I]^2 + \\
&- 2C_{i,I-i+1} \, Cov\left(\Pi_{j=I-i}^{J-1} f_j, \Pi_{j=I-i+1}^{J-1} \hat{f}_j^{I+1}\right) = \\
&= E[C_{iJ} \mid D_I]^2 \theta_{I-i}^2 + \\
&+ E\left[C_{i,I-i+1}^2 \mid D_I\right] \Pi_{j=I-i+1}^{J-1} E\left[\left(\hat{f}_j^{I+1}\right)^2 \mid D_I\right] - E[C_{i,I-i+1} \mid D_I]^2 \, \Pi_{j=I-i+1}^{J-1} E\left[\hat{f}_j^{I+1} \mid D_I\right]^2 + \\
&- 2\,Var[C_{i,I-i+1} \mid D_I] \Pi_{j=I-i+1}^{J-1} E\left[\hat{f}_j^{I+1} \mid D_I\right] f_j = \\
&= E[C_{iJ} \mid D_I]^2 \, \theta_{I-i}^2 + \\
&+ \left(f_{I-i}^2 C_{i,I-i}^2 + \sigma_{I-i}^2 C_{i,I-i}\right) \Pi_{j=I-i+1}^{J-1} E\left[\left(\hat{f}_j^{I+1}\right)^2 \mid D_I\right] - f_{I-i}^2 C_{i,I-i}^2 \, \Pi_{j=I-i+1}^{J-1} E\left[\hat{f}_j^{I+1} \mid D_I\right]^2 +
\end{aligned}$$

$$
\begin{aligned}
&- \; 2\sigma_{I-i}^2 \, C_{i,I-i} \, \Pi_{j=I-i+1}^{J-1} \, E\left[\hat{f}_j^{I+1} \mid D_I\right] f_j = \\
&= \; E[C_{iJ} \mid D_I]^2 \; \theta_{I-i}^2 + \\
&+ \; f_{I-i}^2 \, C_{i,I-i}^2 \, Var\left[\hat{f}_j^{I+1} \mid D_I\right] + \\
&+ \; \sigma_{I-i}^2 \, C_{i,I-i}\left(\Pi_{j=I-i+1}^{J-1} E\left[\left(\hat{f}_j^{I+1}\right)^2 \mid D_I\right] - 2\Pi_{j=I-i+1}^{J-1} E\left[\hat{f}_j^{I+1} \mid D_I\right]^2\right).
\end{aligned} \tag{34}
$$

At this stage using the properties b) and e) of the Lemma 2.5 we can obtain the estimates of the conditional moments:

$$
\begin{aligned}
&\hat{E}\left[\hat{f}_j^{I+1} \mid D_I\right] && = \hat{f}_j^I \\
&\hat{E}\left[\left(\hat{f}_j^{I+1}\right)^2 \mid D_I\right] && = \left(\hat{f}_j^I\right)^2 + \frac{\hat{\sigma}_j^2 C_{I-i,j}}{\left(s_j^{I+1}\right)^2} = \left(\hat{f}_j^I\right)^2 + \frac{\hat{\sigma}_j^2 v_j^2}{C_{I-j,j}} \\
&\overline{Var}\left[\hat{f}_j^{I+1} \mid D_I\right] && = \frac{\hat{\sigma}_j^2 v_j^2}{C_{I-j,j}}
\end{aligned} \tag{35}
$$

where $v_j = C_{I-j,j} / S_j^{I+1}$ is the credibility coefficient as proposed in [Merz and Wüthrich (2015)]. Replace the equations (35) into (34) we can derive an approximation of the process error φ_{iJ}^I:

$$
\begin{aligned}
\hat{\varphi}_{iJ}^I \; &= \left(\hat{C}_{iJ}^I\right)^2 \hat{\theta}_{I-i}^2 + \left(\hat{f}_{I-i}^I\right)^2 C_{i,I-i}^2 \, \Pi_{j=I-i}^{J-1} \frac{\hat{\sigma}_j^2 v_j^2}{C_{I-j,j}} + \\
&+ \; \hat{\sigma}_{I-i}^2 C_{i,I-i} \left(\Pi_{j=I-i+1}^{J-1}\right)\left(\left(\hat{f}_j^I\right)^2 + \frac{\hat{\sigma}_j^2 v_j^2}{C_{I-i,j}}\right) - 2\left(\Pi_{j=I-i+1}^{J-1}\left(\hat{f}_j^I\right)^2\right) + \\
&+ \; \left(\hat{f}_{I-i}^I\right)^2 C_{i,I-i}^2 \Pi_{j=I-i+1}^{J-1}\left(\hat{f}_j^I\right)^2 - \left(\hat{f}_{I-i}^I\right)^2 C_{i,I-i}^2 \Pi_{j=I-i+1}^{J-1}\left(\hat{f}_j^I\right)^2 = \\
&= \left(\hat{C}_{iJ}^I\right)^2 \hat{\theta}_{I-i}^2 + \left(\left(\hat{f}_{I-i}^I\right)^2 C_{i,I-i}^2 + \hat{\sigma}_{I-i}^2 C_{i,I-i}\right)\Pi_{j=I-i+1}^{J-1}\left(\left(\hat{f}_j^I\right)^2 + \frac{\hat{\sigma}_j^2 v_j^2}{C_{I-j,j}}\right) +
\end{aligned}
$$

$$- \ 2\left(\hat{C}_{iJ}^{I}\right)^{2}\hat{\theta}_{I-i}^{2}-\left(\hat{C}_{iJ}^{I}\right)^{2}=$$

$$=\left(\left(\hat{f}_{I-i}^{I}\right)^{2}C_{i,I-i}^{2}+\hat{\sigma}_{I-i}^{2}C_{i,I-i}\right)\Pi_{j=I-i+1}^{J-1}\left(\hat{f}_{j}^{I}\right)^{2}\left(1+\frac{\left(\hat{f}_{j}^{I}\right)^{2}/\hat{\sigma}_{j}^{2}}{C_{I-j,j}}v_{j}^{2}\right)+$$

$$-\ \left(\hat{C}_{iJ}^{I}\right)^{2}\left(1+\hat{\theta}_{I-i}^{2}\right)=$$

$$=\left(\hat{C}_{iJ}^{I}\right)^{2}\left(1+\hat{\theta}_{I-i}^{2}\right)\left(\Pi_{j=I-i+1}^{J-1}\left(1+\hat{\theta}_{I-i}^{2}v_{j}^{2}\right)-1\right)\approx$$

$$\approx\left(\hat{C}_{iJ}^{I}\right)^{2}\Sigma_{j=I-i+1}^{J-1}\hat{\theta}_{j}^{2}v_{j}^{2}=$$

$$=\left(\hat{C}_{iJ}^{I}\right)^{2}\hat{\Phi}_{iJ}, \tag{36}$$

where $\hat{\Phi}_{iJ}=\Sigma_{j=I-i+1}^{J-1}\hat{\theta}_{j}^{2}\,v_{j}^{2}$.

Then we calculate the *parameter error* of (33) using the conditional resamplig technique. By properties b) and c) of the Lemma 2.5 results:

$$\begin{aligned}E\left[\widehat{CDR}_i(I+1)\mid D_I\right] &= E\left[\hat{C}_{iJ}^{I}-\hat{C}_{iJ}^{I+1|D_I}\right]=\\ &= C_{i,I-i}\left(\Pi_{j=I-i}^{J-1}\hat{f}_{j}^{I}-f_{I-i}\Pi_{j=I-i+1}^{J-1}E\left[\hat{f}_{j}^{I+1}\mid D_I\right]\right)=\\ &= C_{i,I-i}\left(\Pi_{j=I-i}^{J-1}\hat{f}_{j}^{I}-f_{I-i}\Pi_{j=I-i+1}^{J-1}\left(\frac{s_{j}^{I}}{s_{j}^{I+1}}\hat{f}_{j}^{I}+f_{j}\frac{C_{I-j,j}}{s_{j}^{I}}\right)\right).\end{aligned} \tag{37}$$

Let be $\alpha_j=S_j^I/S_j^{I+1}$ and therefore $1-\alpha_j=C_{I-j,j}/S_j^{I+1}$ the equation (37) can be

$$E\left[\widehat{CDR}_i(I+1)\mid D_I\right]=C_{i,I-i}\left(\prod_{j=I-i}^{J-1}\hat{f}_{j}^{I}-f_{I-i}\prod_{j=I-i+1}^{J-1}\left(\alpha_j\hat{f}_{j}^{I}+f_j(1-\alpha_j)\right)\right). \tag{38}$$

Now we can derive the parameter error of CDR:

$$E\left[E\left[\widehat{CDR}_i(I+1)\right]^2 \mid D_I\right]=$$

$$= C_{i,I-i}^2 E\left[\left(\Pi_{j=I-i}^{J-1}\hat{f}_j^{J-1} - f_{I-i}\Pi_{j=I-i+1}^{J-1}\left(\alpha_j \hat{f}_j^I + f_j(1-\alpha_j)\right)\right)^2 \mid D_I\right]=$$

$$= C_{i,I-i}^2\left(\Pi_{j=I-i}^{J-1}E\left[\left(\hat{f}_j^I\right)^2 \mid D_I\right]+ f_{I-i}^2\Pi_{j=I-i+1}^{J-1}E\left[\left(\alpha_j\hat{f}_j^I + f_j(1-\alpha_j)\right)^2 \mid D_I\right]+\right.$$

$$\left.- 2E\left[\hat{f}_{I-i}^I f_{I-i}\Pi_{j=I-i+1}^{J-1}\hat{f}_j^I\left(\alpha_j\hat{f}_j^I + f_j(1-\alpha_j)\right) \mid D_I\right]\right)=$$

$$= C_{i,I-i}^2\left(\Pi_{j=I-i}^{J-}\left(Var\left[\hat{f}_j^I \mid D_I\right]+ f_j^2\right)+ f_{I-i}^2\Pi_{j=I-i+1}^{J-1} E\left[\left(\alpha_j\left(\hat{f}_j^I - f_j\right)+ f_j\right)^2\right]+\right.$$

$$\left.- 2E\left[\hat{f}_{I-i}^I f_{I-i}\Pi_{j=I-i+1}^{J-1}\hat{f}_j^I\left(\alpha_j\left(\hat{f}_j^I - f_j\right)+ f_j\right) \mid D_I\right]\right)=$$

$$= C_{i,I-i}^2\left(\Pi_{j=I-i}^{J-1}\left(Var\left[\hat{f}_j^I \mid D_I\right]+ f_j^2\right)+ f_{I-i}^2\Pi_{j=I-i+1}^{J-1}\left(\alpha_j^2 Var\left[\hat{f}_j^I \mid D_I\right]+ f_j^2\right)+\right.$$

$$\left.- 2f_{I-i}^2\Pi_{j=I-i}^{J-1}\left(\alpha Var\left[\hat{f}_j^I \mid D_I\right]+ f_j^2\right)\right)=$$

$$= C_{i,I-i}^2\left(\Pi_{j=I-i}^{J-1}\left(\frac{\sigma_j^2}{s_j^I}+ f_j^2\right)+ f_{I-i}^2\Pi_{j=I-i+1}^{J-1}\left(\alpha_j^2\frac{\sigma_j^2}{s_j^I}+ f_j^2\right)+\right.$$

$$\left.-2f_{I-i}^2\Pi_{j=I-i+1}^{J-1}\left(\alpha_j\frac{\alpha_j^2}{s_j^I}+ f_j^2\right)\right)=$$

$$= C_{i,I-i}^2\left(\Pi_{j=I-i}^{J-1}f_j^2\left(\frac{\alpha_j^2/f_j^2}{s_j^I}+1\right)+ f_{I-i}^2\Pi_{j=I-i+1}^{J-1}f_j^2\left(\alpha_j^2\frac{\sigma_j^2/f_j^2}{s_j^I}+1\right)+\right.$$

$$\left.-2f_{I-i}^2\Pi_{j=I-i+1}^{J-1}f_j^2\left(\alpha_j\frac{\sigma_j^2/f_j^2}{s_j^I}+1\right)\right)=$$

$$= C_{i,I-i}^2\Pi_{j=I-i}^{J-1}f_j^2\left(\Pi_{j=I-i}^{J-1}\left(\eta_j^2+1\right)+\Pi_{j=I-i+1}^{J-1}\left(\alpha_j^2\eta_j^2+1\right)+\right.$$

$$\left.-2\Pi_{j=I-i+1}^{J-1}\left(\alpha_j\eta_j^2+1\right)\right)=$$

$$= C_{i,I-i}^2 \Pi_{j=I-i}^{J-1} f_j^2 \left(\eta_j^2 + 1 + \Pi_{j=I-i+1}^{J-1}\left(\eta_j^2 + 1\right) + \Pi_{j=I-i+1}^{J-1}\left(\alpha_j^2 \eta_j^2 + 1\right) +\right.$$

$$\left. -2\Pi_{j=I-i+1}^{J-1}\left(\alpha_j \eta_j^2 + 1\right)\right) \approx$$

$$\approx C_{i,I-i}^2 \Pi_{j=I-i}^{J-1} f_j^2 \left(\eta_j^2 + 1 + \Sigma_{j=I-i+1}^{J-1} \eta_j^2 + 1 + \Sigma_{j=I-i+1}^{J-1} \alpha_j^2 \eta_j^2 + 1 +\right.$$

$$\left. -2\left(\Sigma_{j=I-i+1}^{J-1} \alpha_j \eta_j^2 + 1\right)\right) =$$

$$= C_{i,J}^2 \left(\eta_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1} (1-\alpha_j)^2 \eta_j^2\right) =$$

$$= \hat{C}_{i,J}^2 \left(\eta_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1} v_j^2 \eta_j^2\right) =$$

where $\eta_j^2 = \dfrac{\sigma_j^2 / f_j^2}{s_j^I}$.

Finally we can get the estimate the parameter error as:

$$\hat{E}\left[\widehat{CDR}_i(I+1) \mid D_I\right] = \hat{C}_{i,J}^2 \left(\hat{\eta}_{I-i}^2 + \sum_{j=I-i+1}^{J-1} v_j^2 \hat{\eta}_j^2\right) = \hat{C}_{i,J}^2 \hat{\Delta}_{iJ}^I,$$

now keep in mind the (36) we can get estimation of mean square error prediction for CDR (retrospective view) in (33) that it is:

$$\widehat{MSEP}_{\widehat{CDR}_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1)\right) = \left(\hat{C}_{i,J}^I\right)^2 \left(\hat{\Phi}_{iJ}^I + \hat{\Delta}_{iJ}^I\right). \tag{41}$$

The previous result will be used in order to estimate the mean square error of prediction of CDR in a *prospective view* through the following decomposition:

$$MSEP_{\widehat{CDR}_i(I+1)|D_I}(0) =$$

$$= E\left[\left(\widehat{CDR}_i - 0\right)^2 \mid D_I\right] =$$

$$= E\left[\left(\widehat{CDR}_i + CDR_i(I+1) - CDR_i(I+1)\right)^2 \mid D_I\right] =$$

$$= MSEP_{\widehat{CDR}_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1)\right) + E\left[CDR_i(I+1)^2\right] =$$

$$= MSEP_{\widehat{CDR}_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1)\right) + Var[CDR_i(I+1) \mid D_I] + \underbrace{E[CDR_i(I+1) \mid D_I]^2}_{=0}, \tag{42}$$

considering that the estimation of the true CDR is:

$$\widehat{Var}[CDR_i(I+1) \mid D_I] = C_{iJ}^2 \hat{\theta}_{I-i}^2 = \hat{C}_{iJ}^2 \hat{\Psi}_i^I. \tag{43}$$

So the estimate of the mean square error of prediction for observed CDR from zero is:

$$\widehat{MSEP}_{\widehat{CDR}_i(I+1)|D_I}(0) = \left(\hat{C}_{iJ}^I\right)^2 \left(\hat{\Delta}_{iJ}^I + \hat{\Phi}_{iJ}^I + \hat{\Psi}_i^I\right) = \left(\hat{C}_{iJ}^I\right)^2 \left(\hat{\Delta}_{iJ}^I + \hat{\Gamma}_{iJ}^I\right) \tag{44}$$

4.2. Mean square error of prediction for overall accidents year

The aim of this sub-section is to derive the one year MSEP for overall generation. As for a single accident year we have the retrospective view (the deviation from true CDR):

$$MSEP_{\Sigma_{i=1}^I \widehat{CDR}_i(I+1)|D_I}\left(\sum_{i=1}^I \widehat{CDR}_i(I+1)\right) = E\left[\left(\sum_{i=I}^I \widehat{CDR}_i(I+1) - \sum_{i=1}^I CDR_i(I+1)\right)^2 \mid D_I\right] \tag{45}$$

and the prospective view (the deviation from zero as required in Solvency 2)

$$MSEP_{\Sigma_{i=1}^I \widehat{CDR}_i(I+1)|D_I}(0) = E\left[\left(\sum_{i=1}^I \widehat{CDR}_i(I+1) - 0\right)^2 \mid D_I\right]. \tag{46}$$

Starting we have to evaluate the correlation among the accident years' estimates provided that the estimators of development factors are used for overall generations in the prediction of ultimate cost. Thus we calculate the mean square error of prediction between two accident years, now let $k > i$:

$$MSEP_{\widehat{CDR}_i(I+1)+\widehat{CDR}_k(I+1)}\left(\widehat{CDR}_i(I+1) + \widehat{CDR}_k(I+1)\right) =$$

$$= E\left[\left(CDR_i(I+1) - \widehat{CDR}_i(I+1) + CDR_k(I+1) - \widehat{CDR}_k(I+1)\right)^2 \mid D_I\right] =$$

$$= MSEP_{\widehat{CDR}_i(I+1)}\left(\widehat{CDR}_i(I+1)\right) + MSEP_{\widehat{CDR}_k(I+1)}\left(\widehat{CDR}_k(I+1)\right) +$$

$$+2E\left[\left(CDR_i(I+1) - \widehat{CDR}_i(I+1)\right)\left(CDR_k(I+1) - \widehat{CDR}_k(I+1)\right) \mid D_I\right] =$$

$$= MSEP_{\widehat{CDR}_i(I+1)}\left(\widehat{CDR}_i(I+1)\right) + MSEP_{\widehat{CDR}_k(I+1)}\left(\widehat{CDR}_k(I+1)\right) +$$

$$+2\underbrace{E[CDR_i(I+1)CDR_k(I+1)\mid D_I]}_{=0}-2E\underbrace{[\widehat{CDR}_i(I+1)CDR_k(I+1)\mid D_I]}_{=0}+$$

$$+2E\left[CDR_i(I+1)\widehat{CDR}_k(I+1)\mid D_I\right]-2E\left[\widehat{CDR}_i(I+1)\widehat{CDR}_k(I+1)\mid D_I\right]=$$

$$=MSEP_{\widehat{CDR}_i(I+1)}\left(\widehat{CDR}_i(I+1)\right)+MSEP_{\widehat{CDR}_k(I+1)}\left(\widehat{CDR}_k(I+1)\right)+$$

$$+\ 2\left(\psi_{ik}^I+E\left[\widehat{CDR}_i(I+1)\mid D_I\right]E\left[\widehat{CDR}_k(I+1)\mid D_I\right]\right)$$

where $\psi_{ik}^I=Cov\left[\widehat{CDR}_i(I+1),\widehat{CDR}_k(I+1)\mid D_I\right]-Cov\left[CDR_i(I+1),\widehat{CDR}_k(I+1)\mid D_I\right]$. In order to obtain an estimation for ψ_{ik}^I we will use again the properties a) and c) of the Lemma 2.5, so we have for $k>i>1$:

$$\begin{aligned}
\psi_{ik}^I &= Cov\left(\widehat{CDR}_i(I+1)-CDR_i(I+1)\mid D_I\right)\\
&= Cov[\hat{C}_{iJ}^{I+1}-E[C_{iJ}\mid D_{I+1}],\hat{C}_{kJ}^{I+1}\mid D_I]=\\
&= E[(\hat{C}_{iJ}^{I+1}-E[C_{iJ}\mid D_I])\,\hat{C}_{kJ}^{I}\mid D_I]-E\,[\hat{C}_{iJ}^{I+1}-E[C_{iJ}\mid D_I]]E[\hat{C}_{kJ}^{I}\mid D_I]=\\
&= E[\hat{C}_{iJ}^{I+1}\,\hat{C}_{kJ}^{I+1}\mid D_I]-E[E[C_{iJ}\mid D_I]\hat{C}_{kJ}^{I}\mid D_I]+\\
&- E[C_{k,I-k+1}\mid D_I]\Pi_{j=I-k+1}^{J-1}E\left[\hat{f}_j^{I+1}\mid D_I\right]C_{i,I-i}f_{I-i}\left(\Pi_{j=I-k+1}^{J-1}E\left[\hat{f}_j^{I+1}\mid D_I\right]-\Pi_{j=I-k+1}^{J-1}\hat{f}_j^{I}\right)=\\
&= E[C_{k,I-k+1}\mid D_I]\Pi_{j=I-k+1}^{I-i-1}E\left[\hat{f}_j^{I+1}\mid D_I\right]E[C_{i,I-i+1}\hat{f}_{I-i}^{I+1}\mid D_I]\Pi_{j=I-i+1}^{J-1}E\left[\left(\hat{f}_j^{I+1}\right)^2\mid D_I\right]+\\
&- E[C_{k,I-k+1}\mid D_I]E\left[C_{i,I-i+1}\Pi_{j=I-k+1}^{J-1}\hat{f}_j^{I+1}\mid D_I\right]\Pi_{j=I-i+1}^{J-1}f_j+\\
&- E[C_{k,I-k+1}\mid D_I]\Pi_{j=I-k+1}^{I-i-1}E\left[\hat{f}_j^{I+1}\mid D_I\right]C_{i,I-i}f_{I-i}\left(\Pi_{j=I-i+1}^{J-1}E\left[\hat{f}_j^{I+1}\mid D_I\right]-\Pi_{j=I-i+1}^{J-1}\hat{f}_j^{I}\right)=\\
&= E[C_{k,I-k+1}\mid D_I]\Pi_{j=I-k+1}^{I-i-1}E\left[\hat{f}_j^{I+1}\mid D_I\right]E[C_{i,I-i+1}\,\hat{f}_{I-i}^{I+1}\mid D_I]\Pi_{j=I-i+1}^{J-1}E\left[\left(\hat{f}_j^{I+1}\right)^2\mid D_I\right]+\\
&- E\,[C_{k,I-k+1}\mid D_I]\Pi_{j=I-k+1}^{J-1}E\left[\hat{f}_j^{I+1}\mid D_I\right]\Pi_{j=I-i+1}^{J-1}f_jE\left[C_{i,I-i+1}\,\hat{f}_{I-i}^{I+1}\right]+\\
&- E[C_{k,I-k+1}\mid D_I]\Pi_{j=I-k+1}^{I-i-1}E\left[\hat{f}_j^{I+1}\mid D_I\right]C_{i,J-i}f_{I-i}\left(\Pi_{j=I-i+1}^{J-1}E\left[\hat{f}_j^{I+1}\mid D_I\right]-\Pi_{j=I-i+1}^{J-1}\hat{f}_j^{I}\right)=\\
&= E[C_{k,I-k+1}\mid D_I]\Pi_{j=I-k+1}^{I-i-1}E\left[\hat{f}_j^{I+1}\mid D_I\right]\times\\
&\times\ E[C_{i,I-i+1}\,\hat{f}_{I-i}^{I+1}\mid D_I]\left(\Pi_{j=I-i+1}^{J-1}E\left[\left(\hat{f}_j^{I+1}\right)^2\mid D_I\right]-\Pi_{j=I-i+1}^{J-1}f_jE\left[\left(\hat{f}_j^{I+1}\right)\mid D_I\right]\right)
\end{aligned}$$

$$- E[C_{i,I-i+1} \mid D_I] E\left[\hat{f}_{I-i}^{I+1} \mid D_I\right] \left(\Pi_{j=I-i+1}^{J-1} E\left[\left(\hat{f}_j^{I+1}\right) \mid D_I\right]^2 - \Pi_{j=I-i+1}^{J-1} f_j E\left[\left(\hat{f}_j^{I+1}\right) \mid D_I\right]\right) \tag{48}$$

now using the properties b), e) and f) of the Lemma 2.5, it is possible to substitute the estimates of the moments conditional to I of the coefficient at time $I+1$ with the estimates of the factors and the variances at time I and the estimation

$$\begin{aligned}
\Psi_{ik}^I &= \hat{C}_{k,I-i}^I \frac{\left(\hat{f}_{I-i}^I\right)^2 C_{i,I-i}^2 + \hat{\sigma}_{I-i}^2 C_{i,I-i} + s_{I-i+1}^{I+1} \hat{f}_{I-i}^I C_{i,I-i}}{s_{I-i}^{I+1}} \left(\Pi_{j=I-i+1}^{J-1} \left((\hat{f}_{I-i}^I)^2 + \frac{\hat{\sigma}_j^2 C_{I-j,j}}{(s_j^{I+1})^2} \right) - \Pi_{j=I-i+1}^{J-1} (\hat{f}_{I-i}^I)^2 \right) = \\
&= \hat{C}_{k,I-i}^I (\hat{f}_{I-i}^I)^2 C_{i,I-i} \left(\hat{v}_{I-i}^2 \frac{C_{i,I-i}}{s_{I-i}^{I+1}} + \frac{\hat{\sigma}_{I-i}^2 / (\hat{f}_{I-i}^I)^2}{s_{I-i}^{I+1}} + \frac{s_{I-i+1}^{I+1}}{s_{I-i}^{I+1} \hat{f}_{I-i}^I} \right) \times \\
&\times \left(\Pi_{j=I-i+1}^{J-1} \left((\hat{f}_{I-i}^I)^2 + \frac{\hat{\sigma}_j^2 C_{I-j,j}}{(s_j^{I+1})^2} \right) - \Pi_{j=I-i+1}^{J-1} (\hat{f}_{I-i}^I)^2 \right) = \\
&= \hat{C}_{k,I-i}^I (\hat{f}_{I-i}^I)^2 C_{i,I-i} \left(\frac{C_{i,I-i}}{\Sigma_{k=0}^i} + \frac{\hat{\sigma}_{I-i}^2 / (\hat{f}_{I-i}^I)^2}{s_{I-i}^{I+1}} + \frac{\Sigma_{k=0}^{i-1} C_{k,I-i+1} \Sigma_{k=0}^{i-1} C_{k,I-i}}{\Sigma_{k=0}^{i} C_{k,I-i} \Sigma_{k=0}^{i-1} C_{k,I-i+1}} \right) \times \\
&\times \left(\Pi_{j=I-i+1}^{J-1} \left(\hat{f}_{I-i}^I \right)^2 \left(1 + \frac{\hat{\sigma}_j^2 / (\hat{f}_{I-i}^I)^2}{(s_j^{I+1})^2} C_{I-j,j} \right) - \Pi_{j=I-i+1}^{J-1} (\hat{f}_{I-i}^I)^2 \right) = \\
&= \hat{C}_{iJ}^I \hat{C}_{kJ}^I \left(1 + \frac{\hat{\sigma}_j^2 / (\hat{f}_{I-i}^I)^2}{(s_j^{I+1})} \right) \left(\Pi_{j=I-i+1}^{J-1} \left(1 + \frac{\hat{\sigma}_j^2 / (\hat{f}_{I-i}^I)^2}{(s_j^{I+1})^2} C_{I-j,j} \right) - 1 \right) = \\
&= \hat{C}_{iJ}^I \hat{C}_{kJ}^I \left(1 + \hat{\zeta}_{I-i}^2 \right) \left(\Pi_{j=I-i+1}^{J-1} \left(1 + \hat{\theta}_j^2 v_j^2 \right) - 1 \right) \approx \\
&\approx \hat{C}_{iJ}^I \hat{C}_{kJ}^I \Sigma_{j=I-i+1}^{J-1} \hat{\theta}_j^2 v_j^2 = \hat{C}_{iJ}^I \hat{C}_{kJ}^I,
\end{aligned} \tag{49}$$

where $\zeta_j^2 = \theta_j^2 v_j$ is the coefficient of variation normalized to the square root basis for calculation of the factors at time $I + 1$.

For the error term between two accident years we always refer to the conditonal resampling tecnique; first we calcolate the following expression with the help of the properties b) and c) of the Lemma 2.5.

$$E\left[\widehat{CDR}_i(I+1)\mid D_I\right]E\left[\widehat{CDR}_k(I+1)\mid D_I\right]=$$

$$= C_{i,I-i}C_{k,I-k}\left(\Pi_{j=I-i}^{J-1}\hat{f}_j^I - f_{I-i}\,\Pi_{j=I-i+1}^{J-1}\left(\frac{s_j^I}{s_j^{I+1}}\hat{f}_j^I + f_j\frac{C_{I-j,j}}{s_j^{I+1}}\right)\right)\times \tag{50}$$

$$\times\left(\Pi_{j=I-k}^{J-1}\hat{f}_j^I - f_{I-k}\,\Pi_{j=I-k+1}^{J-1}\left(\frac{s_j^I}{s_j^{I+1}}\hat{f}_j^I + f_j\frac{C_{I-j,j}}{s_j^{I+1}}\right)\right)$$

now we write the expectation of (50) as:

$$E\left[E\left[\widehat{CDR}_i(I+1)\mid D_I\right]E\left[\widehat{CDR}_k(I+1)\mid D_I\right]\right]=C_{i,I-i}C_{k,I-k}H_i\prod_{j=I-k}^{I-i-1}f_j \tag{51}$$

where H_i is:

$$H_i = \Pi_{j=I-i}^{J-1}E\left[\left(\hat{f}_j^I\right)^2\mid D_I\right]+f_{I-i}^2\Pi_{j=I-i+1}^{J-1}E\left[\left(\frac{s_j^I}{s_j^{I+1}}\hat{f}_j^I+f_j\frac{C_{I-j,j}}{s_j^{I+1}}\right)^2\mid D_I\right]-$$

$$- f_{I-i}^2\Pi_{j=I-i+1}^{J-1}E\left[\left(\frac{s_j^I}{s_j^{I+1}}\hat{f}_j^I+f_j\frac{C_{I-j,j}}{s_j^{I+1}}\right)\mid D_I\right]-\Pi_{j=I-i+1}^{J-1}E\left[\left(\frac{s_j^I}{s_j^{I+1}}\hat{f}_j^I+f_j\frac{C_{I-j,j}}{s_j^{I+1}}\right)^2\mid D_I\right]=$$

$$= \Pi_{j=I-i}^{J-1}\left(\frac{\sigma_j^2}{s_j^I}+f_j^2\right)-f_{I-i}^2\Pi_{j=I-i}^2\left(\frac{s_j^I}{s_j^{I+1}}\frac{\sigma_j^2}{s_j^I}+f_j^2\right)+$$

$$- f_{I-i}^2\Pi_{j=I-i}^2\left(\frac{s_j^I\sigma_j^2}{s_j^{I+1}s_j^I}+f_j^2\right)-\Pi_{j=I-i}^2\left(\frac{s_j^I\sigma_j^2}{s_j^{I+1}s_j^I}+f_j^2\right)=$$

$$= \Pi_{j=I-i}^{J-1}f_j^2\left(\Pi_{j=I-i}^{J-1}\left(\eta_j^2+1\right)+\Pi_{j=I-i}^{J-1}\left(\alpha_j^2\eta_j^2+1\right)-\Pi_{k=I=i}^{J-1}\left(\alpha_j\eta_j^2+1\right)-\right.$$

$$\left.- \Pi_{j=I-i}^{J-1}\left(\alpha_j\eta_j^2+1\right)\right)\approx$$

$$\approx \Pi_{j=I-i}^{J-1}f_j^2\left(\Sigma_{j=I-i}^{J-1}\eta_j^2+\Sigma_{j=I-i}^{J-1}\alpha_j^2\eta_j^2+\Sigma_{j=I-i}^{J-1}\alpha_j\eta_j^2-\Sigma_{j=I-i}^{J-1}\alpha_j\eta_j^2\right)=$$

$$= \Pi_{j=I-i}^{J-1}f_j^2\left(\eta_{I-i}^2-\alpha_{I-i}\eta_{I-i}^2\Sigma_{j=I-i+1}^{J-1}(1-\alpha_j)^2\eta_j^2\right)=$$

$$= \Pi_{j=I-i}^{J-1}f_j^2\left(v_{I-i}\eta_{I-i}^2+\Sigma_{j=I-i+1}^{J-1}v_j^2\eta_j^2\right)= \tag{52}$$

with $\Lambda_{iJ} = v_{I-i}\eta_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1} v_j^2\eta_j^2$.

Finally the expectation in (51) is:

$$E\left[E\left[\widehat{CDR}_i(I+1)\mid D_I\right]E\left[\widehat{CDR}_k(I+1)\mid D_I\right]\right] = C_{i,I-i}C_{k,I-k}\prod_{I-k}^{I-i-1} f_j \prod_{j=I-i}^{J-1} f_j^2\Lambda_{iJ} \quad (53)$$

and its estimate:

$$\hat{E}\left[E\left[\widehat{CDR}_i(I+1)\mid D_I\right]E\left[\widehat{CDR}_k(I+1)\mid D_I\right]\right] == C_{i,I-i}C_{k,I-k}\prod_{I-k}^{I-i-1} \hat{f}_j \prod_{j=I-i}^{J-1} \hat{f}_j^2\hat{\Lambda}_{iJ} = \hat{C}_{iJ}^I\hat{C}_{kJ}^I\hat{\Lambda}_{iJ} \quad (54)$$

with $\hat{\Lambda}_{iJ} = v_{I-i}\,\hat{\eta}_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1} v_j^2\hat{\eta}_j^2$.

The estimation of the mean square of prediction in (45) for the observed CDR

from the true CDR is:

$$\begin{aligned}&\widehat{MSEP}_{\Sigma_{i=1}^I \widehat{CDR}_i(I+1)|D_I}\left(\Sigma_{i=1}^I \widehat{CDR}_i(I+1)\right) = \\ &= \Sigma_{i=1}^I \widehat{MSEP}_{\widehat{CDR}_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1)\right) + 2\Sigma_{k>i>0}\,\hat{C}_{iJ}^I\hat{C}_{kJ}^I\left(\hat{\Lambda}_{iJ} + \hat{\Phi}_{iJ}\right).\end{aligned} \quad (55)$$

The mean square error of prediction for overall generation trought the decomposition of (46) in the following way:

$$MSEP_{\Sigma_{i=1}^I \widehat{CDR}_i(I+1)|D_I}(0) = Var\left[\left(\sum_{i=1}^I \widehat{CDR}_i(I+1)\right)\mid D_I\right] + E\left[\left(\sum_{i=1}^I \widehat{CDR}_i(I+1)\right)\mid D_I\right]^2 \quad (56)$$

the total variance can be decomposed in the classical manner:

$$Var\left[\left(\sum_{i=1}^I \widehat{CDR}_i(I+1)\right)\mid D_I\right] = \sum_{i=1}^I Var\left[\widehat{CDR}_i(I+1)\mid D_I\right] + 2\sum_{k>i} Cov\left[\widehat{CDR}_i(I+1), \widehat{CDR}_k(I+1)\mid D_I\right] \quad (57)$$

keeping in mind that $Var\left[\widehat{CDR}_i(I+1)\mid D_I\right] = Var\left[\hat{C}_{iJ}^{I+1}\mid D_I\right]$ we obtain the following expression:

$$= E\left[C_{i,I-i+1}^2 \mid D_I\right]\Pi_{j=I-i+1}^{J-1}E\left[\left(\hat{f}_j^{I+1}\right)^2 \mid D_I\right] - E[C_{i,I-i+1} \mid D_I]^2\,\Pi_{j=I-i+1}^{J-1}E\left[\left(\hat{f}_j^{I+1}\right) \mid D_I\right]^2 =$$

$$= \left(f_{I-i}^2 C_{i,I-i}^2 + \sigma_{I-i}^2 C_{i,I-i}\right)\Pi_{j=I-i+1}^{J-1}E\left[\left(\hat{f}_j^{I+1}\right)^2 \mid D_I\right] - f_{I-i}^2 C_{i,I-i}^2 \Pi_{j=I-i+1}^{J-1} E\left[\left(\hat{f}_j^{I+1}\right) \mid D_I\right]^2 \tag{58}$$

and its estimation is:

$$\widehat{Var}\left[\hat{C}_{iJ}^{I+1} \mid D_I\right] =$$

$$= \left((\hat{f}_{I-i}^I)^2 C_{i,I-i}^2 + \sigma_{I-i}^2 C_{i,I-i}\right)\left(\Pi_{j=I-i+1}^{J-1}(f_j^I)^2 + \frac{\hat{\sigma}_j^2 C_{I-j,j}}{(S_j^{I+1})^2}\right) - (f_{I-i}^I)^2 C_{i,I-i}^2 \Pi_{j=I-i+1}^{J-1}(\hat{f}_j^I)^2 =$$

$$= (f_{I-i}^I)^2 C_{i,I-i}^2 \left(1 + \frac{\hat{\sigma}_{I-i}^2/(f_{I-i}^I)^2}{C_{i,J-i}}\right)\Pi_{j=I-i+1}^{J-1}(\hat{f}_j^I)^2\left(1 + \frac{\hat{\sigma}_j^2/(f_j^I)^2}{(S_j^{I+1})^2}C_{I-j,j}\right) =$$

$$= \left(\hat{C}_{iJ}^I\right)^2\left((1+\hat{\theta}_{I-i}^2)\Pi_{j=I-i+1}^{J-1}\left(1+\hat{\theta}_j^2 v_j^2\right) - 1\right) \approx$$

$$\approx \left(\hat{C}_{iJ}^I\right)^2\left(\hat{\theta}_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1}\hat{\theta}_j^2 v_j^2 + \underbrace{\hat{\theta}_{I-i}^2 \sum_{j=I-i+1}^{J-1}\hat{\theta}_j^2 v_j^2}_{\approx 0}\right) \approx$$

$$\approx \left(\hat{C}_{iJ}^I\right)^2\left(\hat{\theta}_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1}\hat{\theta}_j^2 v_j^2\right) =$$

$$= \left(\hat{C}_{iJ}^I\right)^2 \hat{\Gamma}_{iJ} \tag{59}$$

while the covariance is:

$$Co\left[C_{iJ}^{I+1}, C_{kJ}^{I+1} \mid D_I\right] =$$

$$= E[C_{k,I-k+1} \mid D_I]\Pi_{j=I-k+1}^{I-i-1} E\left[\hat{f}_j^{I+1} \mid D_I\right]E[C_{i,I-i+1}\hat{f}_{I-i}^{I+1} \mid D_I]\Pi_{j=I-i+1}^{J-1}E\left[\left(\hat{f}_j^{I+1}\right)^2 \mid D_I\right] +$$

$$- E[C_{k,I-k+1} \mid D_I]\Pi_{j=I-k+1}^{J-1} E\left[\hat{f}_j^{I+1} \mid D_I\right]E[C_{i,I-i+1} \mid D_I]\Pi_{j=I-i+1}^{J-1}E\left[\hat{f}_j^{I+1} \mid D_I\right] =$$

$$= E[C_{k,I-k+1} \mid D_I]\Pi_{j=I-k+1}^{J-1} E\left[\hat{f}_j^{I+1} \mid D_I\right] \times$$

$$\times\left(E\left[C_{i,I-i+1}\hat{f}_{I-i}^{I+1}\mid D_I\right]\Pi_{j=I-i+1}^{J-1}E\left[\left(\hat{f}_j^{I+1}\right)^2\mid D_I\right]-E[C_{i,I-i+1}\mid D_I]E[\hat{f}_{I-i}^{I+1}\mid D_I]\Pi_{j=I-i+1}^{J-1}E\left[\left(\hat{f}_j^{I+1}\right)\mid D_I\right]^2\right.\tag{60}$$

and the estimation is:

$$\begin{aligned}
&\widehat{Cov}\left[\hat{C}_{iJ}^{I+1},\hat{C}_{kJ}^{I+1}\mid D_I\right]=\\
&=\hat{C}_{k,I-i}\left(\hat{f}_{I-i}^{I}\right)^2C_{i,I-i}\left(\left(1+\frac{\hat{\sigma}_{I-i}^2/\left(\hat{f}_{I-i}^{I}\right)^2}{S_{I-i}^{I+1}}\right)\Pi_{j=I-i+1}^{J-1}\left((\hat{f}_j^I)^2+\frac{\hat{\sigma}_j^2C_{I-j,j}}{(S_j^{I+1})^2}\right)-\Pi_{j=I-i+1}^{J-1}\left(\hat{f}_j^I\right)^2\right)\\
&=\hat{C}_{kJ}\hat{C}_{iJ}\left(\left(1+\hat{\zeta}_{I-i}^2\right)\Pi_{j=I-i+1}^{J-1}\left(1+\hat{\zeta}_j^2v_j\right)-1\right)\approx\\
&\approx C_{kJ}\hat{C}_{iJ}\left(\hat{\zeta}_{I-i}^2+\Sigma_{j=I-i+1}^{J-1}\hat{\zeta}_{I-i}^2\,v_j\right)=\\
&=\hat{C}_{kJ}\hat{C}_{iJ}\left(\hat{\theta}_{I-i}^2v_{I-i}+\Sigma_{j=I-i+1}^{J-1}\hat{\theta}_{I-i}^2v_j^2\right)=\\
&=\hat{C}_{kJ}\hat{C}_{iJ}\hat{\Xi}_{iJ}^{I}
\end{aligned}\tag{61}$$

the second term of (56) can estimate as:

$$\begin{aligned}
\hat{E}\left[E\left[\left(\Sigma_{i=1}^{I}\widehat{CDR}_i(I+1)\right)\mid D_I\right]^2\right]&=\Sigma_{i=1}^{I}\hat{E}\left[E\left[\left(\widehat{CDR}_i(I+1)\right)\mid D_I\right]^2\right]+\\
&+2\Sigma_{k>i}\underbrace{\hat{E}\left[E\left[\widehat{CDR}_i(I+1)\mid D_I\right]E\left[\widehat{CDR}_k(I+1)\mid D_I\right]\right]}_{\hat{C}_{kJ}\hat{C}_{iJ}\hat{\Lambda}_{iJ}}.
\end{aligned}\tag{62}$$

Finally the estimation of mean square error of prediction for CDR from zero for overall generation is:

$$MSEP_{\Sigma_{i=1}^{I}\widehat{CDR}_i(I+1)|D_I}(0)=\sum_{i=1}^{I}M\hat{S}EP_{\widehat{CDR}_i(I+1)|D_I}(0)+2\sum_{k>i}\hat{C}_{kJ}\hat{C}_{iJ}\left(\hat{\Lambda}_{iJ}+\hat{\Xi}_{iJ}^{I}\right).\tag{63}$$

5. THE GREEKS

In order to calculate in a convenient way the MSEP of CDR it is useful calculate the Greeks used in the previous sections. We distingued between *small Greeks* and *Capital Greeks*. Then the MSEP can be estimated both *small Greeks* and *Capital Greeks* with tailor made formula. For better understanding we resume the Greeks:

- $\hat{\eta}_j^2 = \dfrac{\hat{\sigma}_j^2 / \hat{f}_j^2}{s_j^I}$;
- $\theta_j^2 = \dfrac{\hat{\sigma} / \hat{f}_j^2}{C_{I-i,j}}$
- $v_j = C_{I-j,j} / S_j^{I+1}$;
- $\hat{\zeta}_j^2 = \dfrac{\hat{\sigma}_j^2 / \hat{f}_j^2}{s_j^{I+1}} = \hat{\theta}_j^2 v_j$

from previous figures we can derive the useful relation $\hat{\eta}_j^2 + \hat{\theta}_j^2 = \dfrac{\hat{\eta}_j^2}{v_j}$, the proof is quite easy and it is the following:

$$\begin{aligned} \hat{\eta}_j^2 + \hat{\theta}_j^2 &= \frac{\hat{\sigma}_j^2}{f_j^2}\left(\frac{1}{S_j^I} + \frac{1}{C_{I-j,j}}\right) = \frac{\hat{\sigma}_j^2}{f_j^2}\left(\frac{S_j^I + C_{I-j,j}}{S_j^I C_{I-j,j}}\right) = \\ &= \frac{\hat{\sigma}_j^2 / \hat{f}_j^2}{S_j^I} \frac{S_j^{I+1}}{C_{I-i,j}} = \frac{\hat{\eta}_j^2}{v_j}. \end{aligned} \tag{64}$$

The code in Listing 3 can be used to calculate the *small Greeks*, while the results for the usual triangle are reported in Table 5

Listing 3: Code for Table 4: Estimation of small Greeks.

```
#The Code 1 and Code 2 must be runned before this.
theta2 <-  (beta^2)/Last.Diag[dev1+1, 'Cij']
theta  <-  sqrt(theta2)
nu     <-  Last.Diag[dev1+1, 'Cij'] / S_I1[dev1+1]
nu2    <-  nu^2
zeta2  <-  theta2*nu
zeta   <-  sqrt(zeta2)
Table.5 <- cbind(eta, theta, zeta, nu)
```

Since θ is connected to the process error decreases more quickly for development year than η and ζ are connected to the parameter error.

Table 5
Estimation of *small Greeks*. The output is made by the code 3

j	$\hat{\eta}_j$	$\hat{\theta}_j$	$\hat{\zeta}_j$	v_j
0	0.0212	0.0864	0.0206	0.0569
1	0.0106	0.0432	0.0103	0.0563
2	0.0066	0.0245	0.0064	0.0677
3	0.0066	0.0235	0.0064	0.0738
4	0.0056	0.0173	0.0054	0.0965
5	0.0040	0.0104	0.0037	0.1264
6	0.0078	0.0178	0.0071	0.1619
7	0.0038	0.0077	0.0034	0.1937
8	0.0038	0.0074	0.0034	0.2077
9	0.0056	0.0094	0.0048	0.2630
10	0.0189	0.0271	0.0155	0.3271
11	0.0086	0.0090	0.0062	0.4779

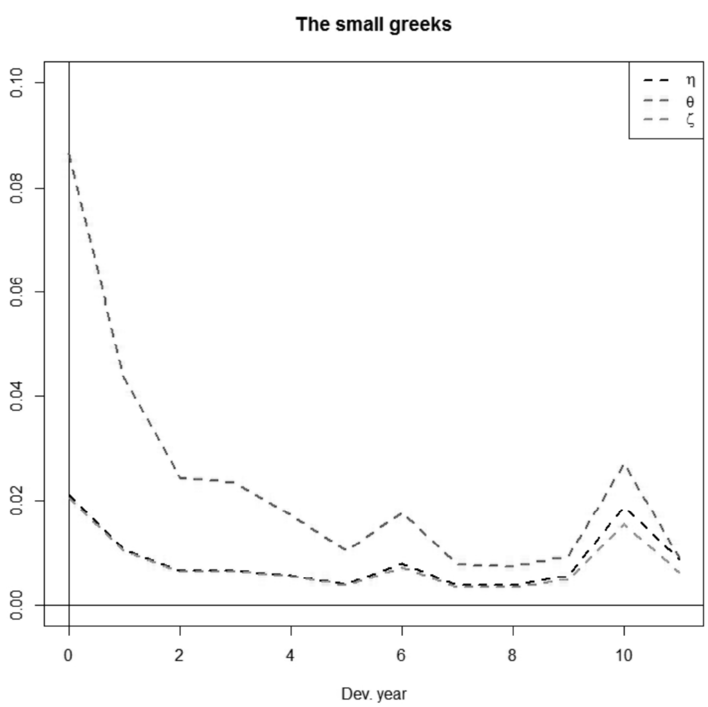

Figure 2: Comparison of *small Greeks* as reported in Table 5

The *Capital Greeks* can be computed through the *small Greeks* in the following way:

- $\hat{\Delta}_{iJ} = \hat{\eta}_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1} v_j^2 \hat{\eta}_j^2$;
- $\hat{\Phi}_{iJ} = \Sigma_{j=I-i+1}^{J-1} v_j^2 \hat{\theta}_j^2$;
- $\hat{\Psi}_i = \theta_{I-i}^2$;
- $\hat{\Gamma}_{iJ} = \hat{\Phi}_{iJ} + \hat{\Psi}_i$;
- $\hat{\Lambda}_{iJ} = v_{I-i}\hat{\eta}_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1} v_j^2 \hat{\eta}_j^2$
- $\hat{\Xi}_{iJ} = \hat{\theta}_{I-i}^2 v_{I-i} + \Sigma_{j=I-i+1}^{J-1} \hat{\theta}_j^2 v_j^2 = \hat{\zeta}_{I-i}^2 + \Sigma_{j=I-i}^{J-1} \hat{\zeta}_j^2 v_j$

In order to estimate the MSEP of the CDR for a single accident year, we can compute the (44) with the following alternative:

$$\begin{aligned}\hat{\Delta}_{iJ} + \hat{\Gamma}_{iJ} &= \hat{\eta}_{I-i}^2 + \theta_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1}\left(\hat{\eta}_j^2 + \hat{\theta}_j^2\right)v_j^2 \\ &= \frac{\hat{\eta}_{I-i}^2}{v_{I-i}} + \Sigma_{j=I-i+1}^{J-1}\hat{\eta}_j^2 v_j,\end{aligned} \tag{65}$$

thus the (44) can written as:

$$\widehat{MSEP}_{\widehat{CDR}_i(I+1)|D_I}(0) = \left(\hat{C}_{iJ}^I\right)^2 \left(\frac{\hat{\eta}_{I-i}^2}{v_{I-i}} + \sum_{j=I-i+1}^{J-1} \hat{\eta}_j^2 v_j\right), \tag{66}$$

in a similar way we obtain $\hat{\Phi}_{iJ} + \hat{\Delta}_{iJ} = \hat{\eta}_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1}\hat{\eta}_j^2 v_j$, and it can be substitute in (41) and it becomes:

$$\widehat{MSEP}_{\widehat{CDR}_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1))\right) = \left(\hat{C}_{iJ}^I\right)^2 \left(\hat{\eta}_{I-i}^2 + \sum_{j=I-i+1}^{J-1} \hat{\eta}_j^2 v_j\right) \tag{67}$$

Also the (63) has an alternative form, so we derive the following relation:

$$\begin{aligned}\hat{\Lambda}_{iJ} + \hat{\Xi}_{iJ} &= v_{I-i}\hat{\eta}_{I-i}^2 + v_{I-i}\hat{\theta}_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1}\left(\hat{\eta}_j + \hat{\theta}_j\right)v_j^2 = \\ &= \hat{\eta}_{I-i}^2 + \Sigma_{j=I-i+1}^{J-1}\hat{\eta}_j^2 v_j.\end{aligned} \tag{68}$$

So the MSEP of CDR from 0 in (63) can be also calculated by means of *small Greeks*:

$$MSEP_{\Sigma_{i=1}^{I}\widehat{CDR}_i(I+1)|D_I}(0)=\sum_{i=1}^{I}M\hat{S}EP_{\widehat{CDR}_i(I+1)|D_I}(0)+2\sum_{k>i}\hat{C}_{kJ}\hat{C}_{iJ}\left(\hat{\eta}_{I-i}^2+\sum_{j=I-i+1}^{J-1}\hat{\eta}_j^2 v_j\right),$$

while for (55) we use $\hat{\Lambda}_{iJ}+\hat{\Phi}_{iJ}=\hat{\eta}_{I-i}^2 v_{I-i}+\Sigma_{j=I-i}^{J-1}\hat{\eta}_j^2 v_j$

$$\widehat{MSEP}_{\Sigma_{i=1}^{I}\widehat{CDR}_i(I+1)|D_I}\left(\Sigma_{i=1}^{I}\widehat{CDR}_i(I+1)\right)=$$
$$=\Sigma_{i=1}^{I}\widehat{MSEP}_{\widehat{CDR}_i(I+1)|D_I}\left(\widehat{CDR}_i(I+1)\right)+2\Sigma_{k>i>0}\hat{C}_{iJ}^{I}\hat{C}_{kJ}^{I}\left(\hat{\eta}_{I-i}^2 v_{I-i}+\Sigma_{j=I-i+1}^{J-1}\hat{\eta}_j^2 v_j\right).$$

(70)

Using the *small Greeks* figured in Table (5), then the computation of MSEP for CDR is very simple via (66), (67), (69) and (70). In Listing 4 are reported the instruction to calculate the MSEP for CDR, both 0 than true CDR. The results are showed in Table

Listing 4: Code for Table 6: Estimation of MSEP with *small Greeks*.

```
#The Code 1, Code 2 and Code 3 must be runned before this.
## equation (66), estimation MSEP of CDR from true CDR
MSEP.CDR.True <- rep(0, n.origin)
MSEP.CDR.True[2:n.origin] <-
(hat_CiJ[2:n.origin]^2)*((eta2)[(n.origin-1):1] +
c(0,cumsum((eta2*nu)[(n.origin-1):2])))

## equation (65), estimation MSEP of CDR from zero
MSEP.CDR.Zero <- rep(0,n.origin)
MSEP.CDR.Zero[2:n.origin] <-
(hat_CiJ[2:n.origin]^2)*((eta2/nu)[(n.origin-1):1] +
c(0,cumsum((eta2*nu)[(n.origin-1):2])))

## equation (69), estimation MSEP of CDR from true CDR
covariance <-
2*sum(hat_CiJ[2:n.origin]*((eta2*nu)[(n.origin-1):1] +
c(0,cumsum((eta2*nu)[(n.origin-1):2])))* c(cumsum(hat_CiJ[n.origin:
3])[(n.origin-2):1],0))

MSEP.CDR.True.Tot <- sum(MSEP.CDR.True) + covariance
```

```
## equation (68), estimation MSEP of CDR from zero
covariance<-2*sum(hat_CiJ[2:n.origin]*((eta2)[(n.origin-1):1]
+
c(0,cumsum((eta2*nu)[(n.origin-1):2])))*
c(cumsum(hat_CiJ[n.origin:3])[(n.origin-2):1],0))
MSEP.CDR.Zero.Tot <-  sum(MSEP.CDR.Zero) + covariance

Table.6 <- rbind(cbind(
       MSEP.CDR.True^0.5,
       MSEP.CDR.True^0.5/hat_R_i,
       MSEP.CDR.Zero^0.5,MSEP.CDR.Zero^0.5/hat_R_i,
       (MSEP.CDR.Zero/MSEP_hatCiJ)^0.5),
       c(MSEP.CDR.True.Tot^0.5,MSEP.CDR.True.Tot^0.5/sum(hat_R_i),
       MSEP.CDR.Zero.Tot^0.5,MSEP.CDR.Zero.Tot^0.5/sum(hat_R_i),
       (MSEP.CDR.Zero.Tot/MSEP_MACK_tot)^0.5
              )
       )
```

Table 6
Estimation of rMSEP with the Merz Wüthrich model.
The output is made by the code 4

i	$rMSEP_{CDR}\,(CDR)$	$rMSEP_{CDR}\,(CDR)/\hat{R}_i$	$rMSEP_{CDR}\,(0)$	$rMSEP_{CDR}\,(0)/\hat{R}_i$	$rMSEP_{CDR}(0)/\hat{C}_{i,J}$
0	0	-	0	-	
1	1,915	10.93%	2,770	15.80%	100.00%
2	4,473	16.56%	7,580	28.05%	95.12%
3	3,338	9.44%	4,059	11.48%	45.75%
4	3,247	7.69%	3,717	8.80%	40.92%
5	3,785	6.37%	4,368	7.35%	42.32%
6	4,269	5.77%	6,599	8.93%	56.52%
7	3,505	4.34%	4,389	5.43%	39.17%
8	3,097	3.81%	4,817	5.93%	44.66%
9	2,650	3.30%	4,926	6.14%	46.79%
10	2,626	2.76%	5,007	5.25%	44.20%
11	2,650	2.51%	7,137	6.76%	56.88%
12	4,163	2.83%	14,772	10.04%	75.49%
Tot.	32,534	3.85%	42,707	5.05%	65.52%

6. CONCLUSIONS

The paper has the aim of revisiting the formulas of the Mack model and the model in [Merz and Wüthrich(2008b)]. The goal is achieved by a different formulation through the small Greeks instead of the Capital Greeks used by [Merz and Wüthrich(2008b)]. The main difference between the two formulations is that the Capital Greeks are defined by accident year while the small Greeks are defined at level of development year, thus allowing to analyze the behavior of volatility for development year. This alternative definition has several practical implications, since the small Greeks do not depend on the volume measure of the single accident year, they can be used to compare the evolution of volatility over time between different lines of business of a same undertaking or a particular line of business among different undertakings. Moreover when for a small business or for a start-up volatility cannot be estimated due to lack of data, the small Greeks can be estimated on a similar line of business (e.g. estimated on the parent company) and used in order to estimate the volatility of CDR. Also for triangles with few development years we can extrapolate the future values of small Greeks so we are able to estimate the volatility taking into account the evolution of payments beyond the last development year of the run-off triangle. The paper also provides the R code to replicate the calculations of small Greeks and MSEP, for ultimate volatility and one-year volatility.

In practice, the model has provided the mean (845,851) and the standard deviation (42,707) of the CDR's one year distribution. Provided that the observed CDR for the next balance year equals to 5,475 we can make a normal distribution assumption and refer CDR to the 55.4% percentile of the distribution highlighting a good performance of the model. Also, following [Merz and Wüthrich (2008b)], comparing the observed CDR with the root of prediction error from the true CDR (32,534) we can conclude that the true CDR could be either positive or negative.

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